Lattices&Codes: Algorithmic Connections and New Constructions

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Agenda

Part I. Intro: Lattices&Codes

Part II. Sieving for codes

Part III (if time). Lattice constructions from codes

Part I

Intro: Lattices & Codes

Lattices&Codes: definitions

$\mathcal{L}$	$\mathcal{C}$
Lattice $\mathcal{L}$ – additive group in $\mathbb{R}^n$	Code $\mathcal{C}$ – additive group in $\mathbb{F}_p^n$
Euclidean metric (ℓ_2)	ℓ_1 - metric
$\|\mathbf{v}\|_2$	$wt(\mathbf{v}) = \|\{i : \mathbf{v}[i] > 0\}\|$- Hamming weight

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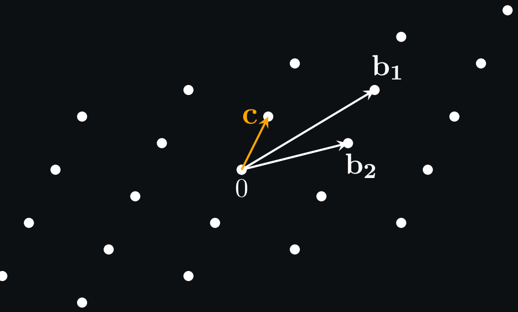
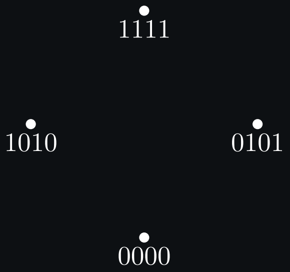
Euclidean metric (ℓ_2)	ℓ_1 - metric
$\|\mathbf{v}\|_2$	$wt(\mathbf{v}) = \lvert\{i : \mathbf{v}[i] > 0\}\rvert$- Hamming weight
$\lambda_1(\mathcal{L})$ - shortest vector	$d(\mathcal{C})$-min. distance
Minkowski bound on $\lambda_1(\mathcal{L})$	Gilbert-Varshamov bound





Lattices&Codes: hard problems

$\mathcal{L}$ | $\mathcal{C}$

Finding a short vector

Given $A \in \mathbb{Z}_q^{(n-k)\times n}$, find $\mathbf{x} \in \mathbb{Z}_q^n$
s.t. $\|\mathbf{x}\| < B$ and $A\mathbf{x} = \mathbf{0} \bmod q$

Given $H \in \mathbb{F}_p^{(n-k)\times n}$, $\mathbf{s} \in \mathbb{F}_p^{n-k}$, find $\mathbf{e} \in \mathbb{F}_p^n$
s.t. $wt(\mathbf{e}) = \omega$ and $H\mathbf{e} = \mathbf{s}$

$$\boxed{A} \; \mathbf{x} = \mathbf{0} \bmod q \qquad \boxed{H} \; \mathbf{e} = \mathbf{s}$$

$$\mathcal{L}^{\perp}(A) = \{\mathbf{x} \in \mathbb{Z}^m \; : \; A\mathbf{x} = \mathbf{0} \bmod q\}$$

aka the SIS problem

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-1

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Algorithms for finding a short vector:	
Enumeration algorithms	ISD algorithms
Sieving for lattice vectors	Sieving for codes[1]

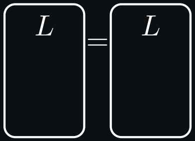
[1] Guo, Q., Johansson, T., Nguyen, V.: A new sieving-style information-set decoding algorithm. Ducas L, Esser A., Etinksi S., Kirshanova E.: Asymptotics and omprovements of sieving for codes

Part II

Sieving for codes

Idea of sieving in lattices

Saturate space with enough lattice vectors so that their sums give short(er) vectors



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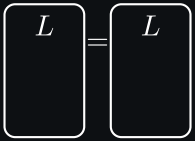

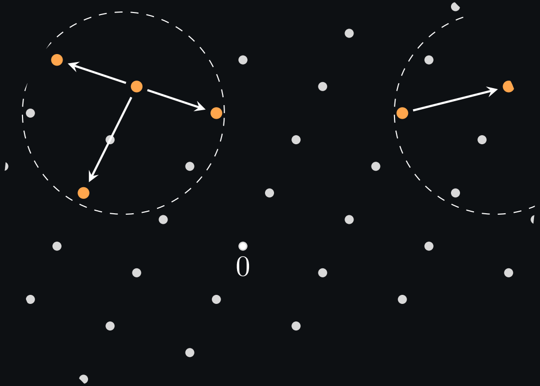

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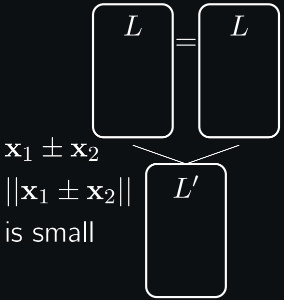

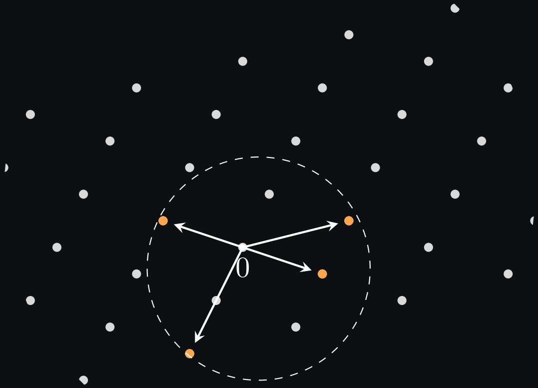

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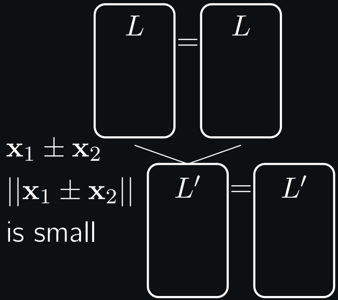

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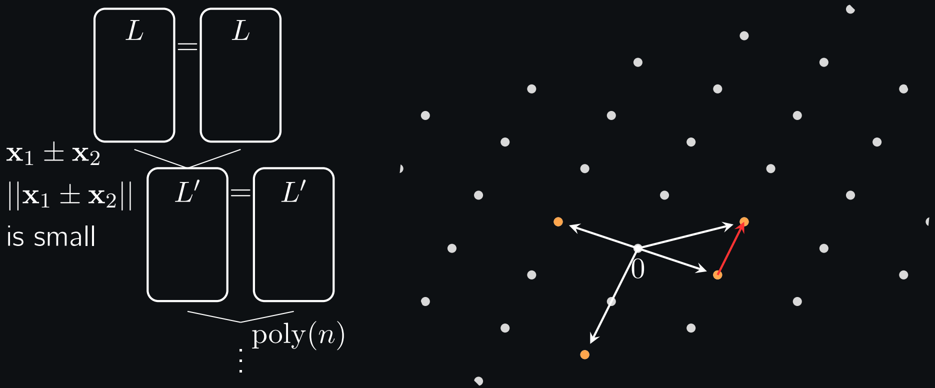

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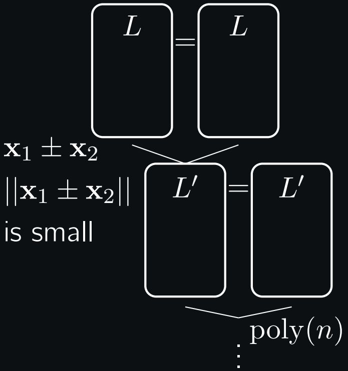
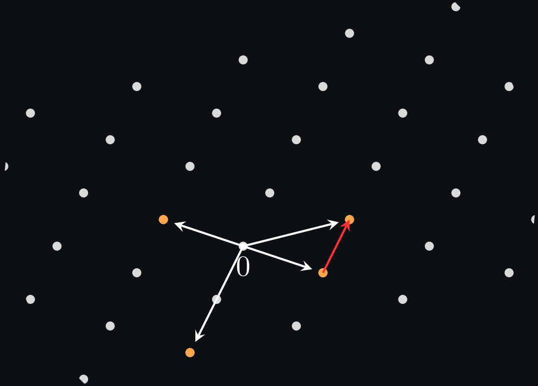
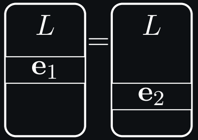




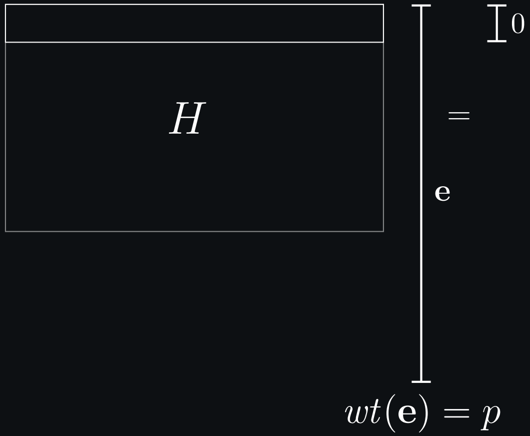
Questions to be addressed: size of L (memory), time to find all close pairs (complexity). Best known algorithm for the short vector problem.

Idea of sieving in binary codes

Keep weight constant, move gradually from subcodes $\mathcal{C}_i := \{\mathbf{e} \ : \ (H\mathbf{e})[0:i] = 0\}$ to the code $\mathcal{C}$: $\mathcal{C}_1 \subset \mathcal{C}_2 \ldots \subset \mathcal{C}$. Choose some weight $p \leq \omega$.



$wt(\mathbf{e}_i) = p$
$(H\mathbf{e}_i)[0] = 0$



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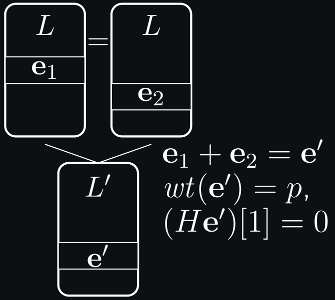

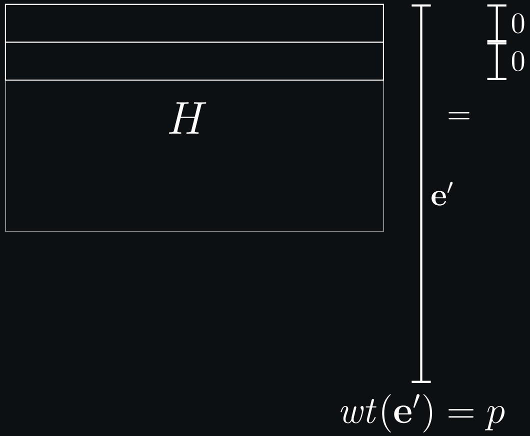

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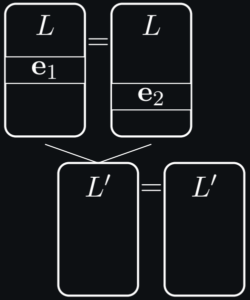

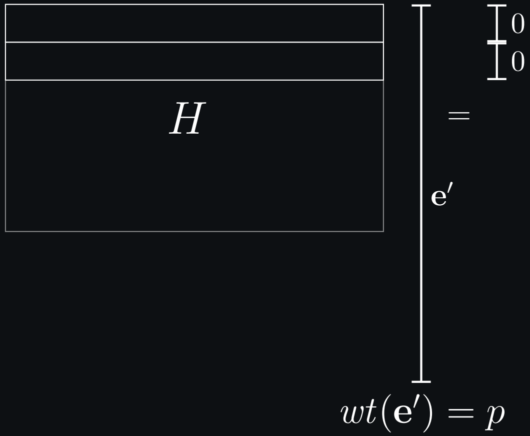

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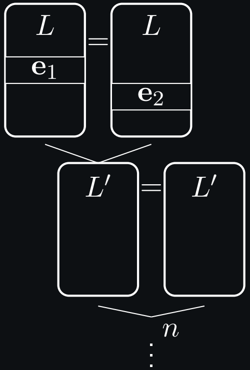
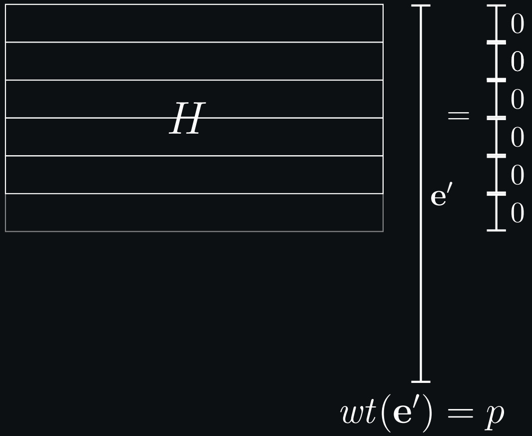
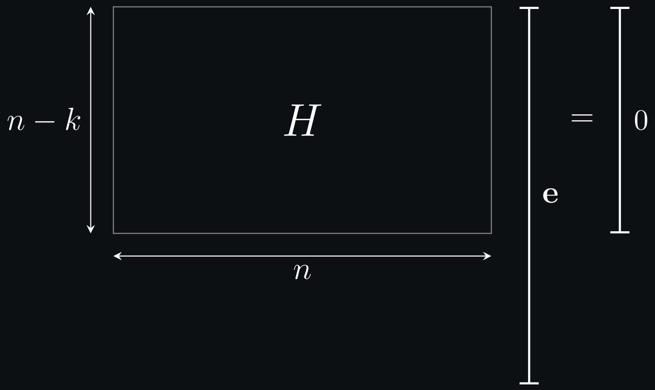




Setting up ISD with Sieving: systematic form



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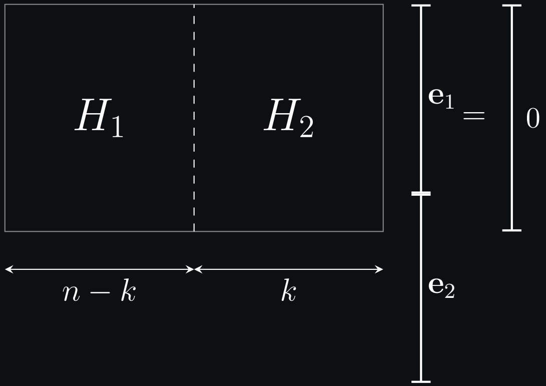

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$$\left[\begin{array}{c|c} \mathbf{I}_{n-k} & H' \end{array}\right] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\underbrace{\hphantom{\mathbf{I}_{n-k}}}_{n-k}\ \underbrace{\hphantom{H'}}_{k}$$

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$$\begin{pmatrix} \underbrace{\mathbf{I}_{n-k-\ell}}_{} & H' \\ \underbrace{0}_{n-k-\ell} & \underbrace{H''}_{k+\ell} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$$

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Sieve for $\mathbf{e}_2$

$$wt(\mathbf{e}_2) = p,\ wt(\mathbf{e}_1) = \omega - p$$

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Sieve for $\mathbf{e}_2$

$$wt(\mathbf{e}_2) = p,\, wt(\mathbf{e}_1) = \omega - p$$

Apply permutations on H until achieve the correct weight distribution on $\mathbf{e}_1, \mathbf{e}_2$.

Sieving for codes: the algorithm

1. Randomly permute H and compute H''.

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3. For $i = 1, \ldots, n$:
   - 3.1 Find all pairs $\mathbf{v}, \mathbf{v} \in L_{i-1}$ with $wt(\mathbf{v} + \mathbf{v}') = p$, store them in L_i
   - 3.2 Discard all $\mathbf{v} \in L_i$ s.t. $\mathbf{v} \notin \mathcal{C}_i$

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Runtime

Success Probability:

$$\underbrace{\Pr[wt(\mathbf{e}'') = p]}_{\frac{\binom{n-k-\ell}{w-p}\binom{k+\ell}{p}}{\binom{n}{w}}} \cdot \underbrace{\Pr[\mathbf{e}'' \in L_n]}_{\frac{N}{\binom{k+\ell}{p}/2^\ell}}$$

Time per iteration (Steps 1–4)

$n \cdot T_{\mathcal{NN}}$

$T_{\mathcal{NN}}$ – runtime of Near Neighbor search (Step 3.1)

Glimpse of the analysis

How large is N?

How large is $T_{\mathcal{N}\mathcal{N}}$?

Glimpse of the analysis

How large is N?

- Want: $\left|\mathbf{w} \in L_i \colon \mathbf{w} \in \mathcal{C}_i\right| \geq N$
- Each new parity-check equation eliminates half of the list elements:

$$\Pr[\mathbf{w} \in \mathcal{C}_i \mid \mathbf{w} \in L_i = \Pr[\mathbf{w} \in \mathcal{C}_i \mid \mathbf{w} \in \mathcal{C}_{i-1}] = \frac{|\mathcal{C}_i|}{|\mathcal{C}_{i-1}|} = 1/2$$

- We want to keep (asymptotically) the same list sizes:

$$\mathbb{E}\big[|\mathbf{w} \in L_i \colon \mathbf{w} \in \mathcal{C}_i|\big] = \mathbb{E}\big[|L_i|\big]/2 \overset{!}{\geq} |L_{i-1}| =: N$$

- $\mathbb{E}\big[|L_i|\big] = |L_{i-1}|^2 \cdot \Pr[wt(\mathbf{v}+\mathbf{v}') = p \colon wt(\mathbf{v}) = wt(\mathbf{v}') = p] =$

$$= N^2 \cdot \frac{\binom{k+\ell}{p} \cdot \binom{p}{p/2}\binom{k+\ell-p}{p/2}}{\binom{k+\ell}{p}^2} \overset{!}{=\geq} 2N \quad \Leftrightarrow \quad N \geq \frac{2\binom{k+\ell}{p}}{\binom{p}{p/2}\binom{k+\ell-p}{p/2}}$$

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How large is $T_{\mathcal{N}\mathcal{N}}$?

Depends on the algorithm...

ISD with Sieving: asymptotics (worst-case rate, GV bound error)

- $wt(\mathbf{v}_1) = wt(\mathbf{v}_2) = wt(\mathbf{v}_1 + \mathbf{v}_2) = p \;\Rightarrow\; wt(\mathbf{v}_1 \wedge \mathbf{v}_2) = p/2$
- Idea: Enumerate potential overlap for each vector

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- Idea: put another random code on top, decode all $\mathbf{v}_i$'s w.r.t. code. Close $\mathbf{v}_i$'s will decode to the same codeword(s).

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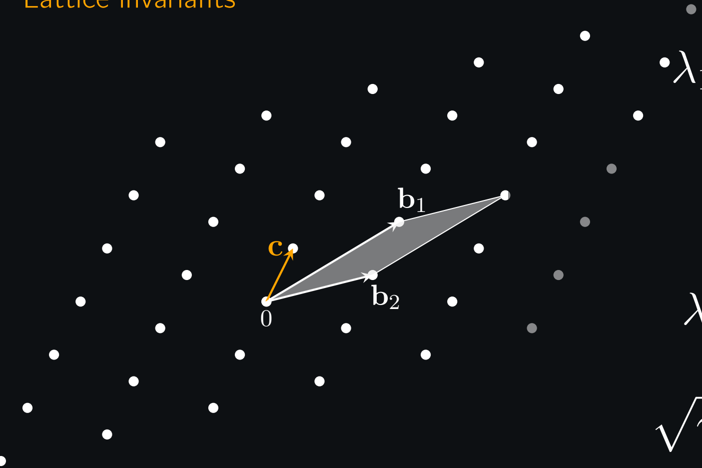
Q?

Part III

From codes to lattices: dense lattice construction

Lattice invariants



Minimum

$$\lambda_1(\Lambda) = \min_{\mathbf{v} \in \Lambda \setminus \mathbf{0}} \|\mathbf{v}\|_2$$

Determinant

$$\det(\Lambda) = |\det(\mathbf{b}_i)_i|$$

Minkowski bound

$$\lambda_1(\Lambda) \leq \sqrt{n} \cdot \det(\Lambda)^{\frac{1}{n}}$$

Normalized min. distance

$$\sqrt{\gamma(\Lambda)} = \lambda_1(\Lambda) / \det(\Lambda)^{\frac{1}{n}}$$

A lattice is a set $\Lambda = \{\sum_{i \leq n} x_i \mathbf{b}_i \ : \ x_i \in \mathbb{Z}\}$ for linearly independent $\mathbf{b}_i \in \mathbb{R}^n$.
$\{\mathbf{b}_i\}_i$ is a basis of Λ

Our goal

$$\sqrt{\gamma(\Lambda)} = \lambda_1(\Lambda) / \det(\Lambda)^{\frac{1}{n}} \leq \sqrt{n}$$

We are interested in

1. explicit construction of a lattice with as large $\gamma(\Lambda)$ as possible
2. with an efficient (list-) decoding algorithm (runtime at most $\mathrm{poly}(n)$).

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Why? We might want to use lattice as codes, hence we care about their decoding properties.

A 'random' lattice (an example will given later) is expected to achieve $\sqrt{\gamma(\Lambda)} \sim \sqrt{n}$, but we do not know how to efficiently decode them.

State-of-the art on $\sqrt{\gamma(\Lambda)}$ ($\Omega()$ for $\sqrt{\gamma(\Lambda)}$ is omitted)

Lattice Λ	$\sqrt{\gamma(\Lambda)}$
Barnes-Wall lattice [BW]	$n^{1/4}$

Defined by the rows of

$$\mathrm{BW}^k = \begin{bmatrix} 1 & 1 \\ 0 & \phi \end{bmatrix}^{\otimes k} \subset \mathbb{C}^{2^k},$$

where $\phi = 1 + i$

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Discrete Logarithm Lattices [DP]	$\frac{\sqrt{n}}{\log n}$

For $(\mathbb{Z}/m\mathbb{Z})^\star$,

p_i – primes, $1 \leq i \leq n$

$\phi : \mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^\star$

$(x_1, \ldots, x_n) \mapsto \prod_{i=1}^n p_i^{x_i}$

$\Lambda_{\mathsf{dlog}} = \ker \phi$.

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Barnes-Wall lattice [BW]	$n^{1/4}$
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Construction-A lattice from Reed-Solomon codes [BP]	$\sqrt{\frac{n}{\log n}}$

Constriction-A:

Take $B \in (\mathbb{Z}/q\mathbb{Z})^{n\times m}$ – a generator matrix of a code.

$\Lambda_{\mathsf{A}} = \mathbb{Z}^n B + q\mathbb{Z}^m \subset \mathbb{Z}^m$

is a construction-A lattice.

State-of-the art on $\sqrt{\gamma(\Lambda)}$ ($\Omega()$ for $\sqrt{\gamma(\Lambda)}$ is omitted)

Lattice Λ	$\sqrt{\gamma(\Lambda)}$
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Lifting sequences of codes to lattices

State-of-the art on $\sqrt{\gamma(\Lambda)}$ ($\Omega()$ for $\sqrt{\gamma(\Lambda)}$ is omitted)

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Construction-A lattice from Reed-Solomon codes [BP]	$\sqrt{\frac{n}{\log n}}$
Construction-D lattice from BCH codes [MP]	$\sqrt{\frac{n}{\log n}}$
Construction-D lattice from subfield subcodes of Garcia-Stichtenoth codes [KM]	$\frac{\sqrt{n}}{(\log n)^{\varepsilon+o(1)}}$

Kirshanova-Malygina'23.
This talk

Main result

Theorem: For a constant $\varepsilon > 0$, there is a family of lattices $\mathcal{L} \subset \mathbb{R}^n$ with normalized minimum distance

$$\frac{\lambda_1(\Lambda)}{\det(\Lambda)^{1/n}} = \Omega\left(\frac{\sqrt{n}}{(\log n)^{\varepsilon+o(1)}}\right).$$

These lattices are list decodable to within distance $\sqrt{1/2} \cdot \lambda_1(\Lambda)$ in $\mathrm{poly}(n)$ time.

Construction-D lattice: simplified definition

- Fix an integer $L \geq 0$, let
$$C_L \subseteq C_{L-1} \subseteq \ldots \subseteq C_1 \subseteq C_0 = \mathbb{F}_p^n$$
be a tower of p-ary codes of length n, where $\dim(C_i) = k_i$.

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- Let $\mathbf{b}_1, \ldots, \mathbf{b}_n$ be a basis of $\mathbb{F}_p^n$ s.t.
$$\mathbf{b}_1, \ldots, \mathbf{b}_{k_i} \text{ is a basis of } C_i \text{ for all } i = 0, \ldots, L.$$
- Define a set of distinguished $\mathbb{Z}^n$ representatives of $\mathbf{c}_i = \sum_{j=1}^{k_i} a_j \mathbf{b}_j \in C_i$ as
$$\overline{\mathbf{c}}_i = \sum_{j=1}^{k_i} \overline{a}_j \overline{\mathbf{b}}_j \in \mathbb{Z}^n \quad \text{where } \overline{a}_j \in \{0, \ldots p-1\} \subset \mathbb{Z}.$$

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- Let $\mathcal{L}_0 = \mathbb{Z}^n$, and for each $i = 1, \ldots, L$ define
$$\Lambda_i = \overline{C}_i + p\Lambda_{i-1}, \quad \overline{C}_i = \{\overline{c}_i \ : \ c_i \in C_i\}.$$
- The construction-D for the tower $\{C_i\}$ is $\Lambda = \Lambda_L$.

Main idea

- Construct a sequence of codes $C_L \subseteq C_{L-1} \subseteq \ldots \subseteq C_1 \subseteq C_0 = \mathbb{F}_q^n$, each C_i is an algebraic-geometric code from a specific function field, the Garcia-Stichtenoth field.
- Such AG-codes are defined over $\mathbb{F}_{p^h}$ for an even h, hence go to subfield-subcodes:

$$C_L \cap \mathbb{F}_p^n \subseteq C_{L-1} \cap \mathbb{F}_p^n \subseteq \ldots C_0 \cap \mathbb{F}_p^n \ = \mathbb{F}_p^n,$$

- We know $\dim(C_i \cap \mathbb{F}_p^n)$ and minimal distance of $C_i \cap \mathbb{F}_p^n$ for all i.
- Compute the minimum $\lambda_1(\Lambda_L)$ of the construction-D lattice Λ_L.
- Compute (an upper bound on) $\det(\Lambda_L)$.
- Conclude on $\gamma(\Lambda) = \lambda_1(\Lambda)/\det(\Lambda_L)$.
- For efficient decoding, adapt soft decision decoding algorithm of Koetter-Vardy.

Conclusions

- Take you favourite code (may be an AG code) with a poly-time decoding algorithm.
- Construct a sequence of codes with a lower bound on min. distance and on dimension.
- These suffice to derive $\lambda_1(\Lambda)$ and (a lower bound) on $\det(\Lambda)$.
- Check if you beat the state-of-the-art.
- Interesting candidate: Bassa-Ritzenthaler towers, (arXiv:1807.05714)