Lower bounds on lattice sieving and information set decoding

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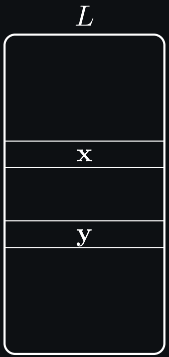
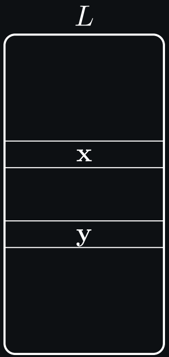
The closest pairs problem



Given a list L and $r > 0$, find almost all $\mathbf{x}, \mathbf{y} \in L$ such that

$$\text{dist}(\mathbf{x}, \mathbf{y}) < r.$$

The closest pairs problem

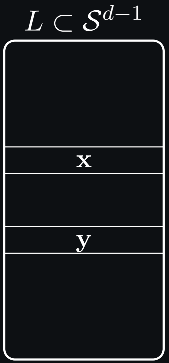


Given a list L and $r > 0$, find almost all $\mathbf{x}, \mathbf{y} \in L$ such that

$$\text{dist}(\mathbf{x}, \mathbf{y}) < r.$$

- Cases of interest:
  - ◇ $L \subset \mathcal{S}^{d-1}$ – a unit sphere, $r = \Theta(1)$
  - ◇ $L \subset \{0, 1\}^d$, $r = \Theta(d)$
- Often $|L| = \exp(d)$ (dense setting)
- Elements in L are uniformly distributed

Why interesting?



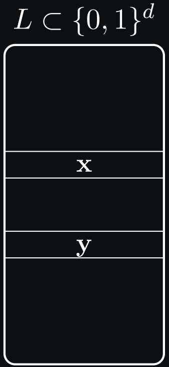
Main subroutine inside sieving algorithms for SVP

For $|L| = \left(\frac{4}{3}\right)^{d/2}$, we can solve this problem in $T = \left(\frac{3}{2}\right)^{d/2}$ time and $S = \left(\frac{4}{3}\right)^{d/2}$ space.

Some NIST submitters use these complexities to setup their parameters.

All $o(d)$ factors in the exponents are omitted

Why interesting?



Main subroutine inside Information Set Decoding algorithms, [MO15], [BM18]

Relevant to the dense error setting: $wt(e) = \Theta(d)$

NIST submitters use sparse error: $wt(e) = o(d)$.

How to solve the closest pairs problem?

Use locality-sensitive hashing (LSH).

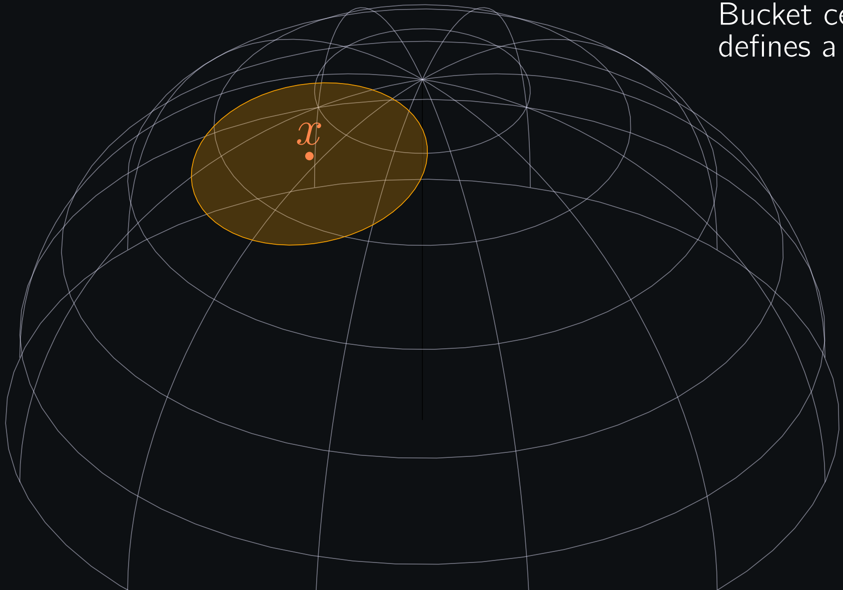
[BGJ15], [BDGL16] for Euclidean metric

[MO15] for Hamming metric

LSH is built upon a family of hash functions h such that

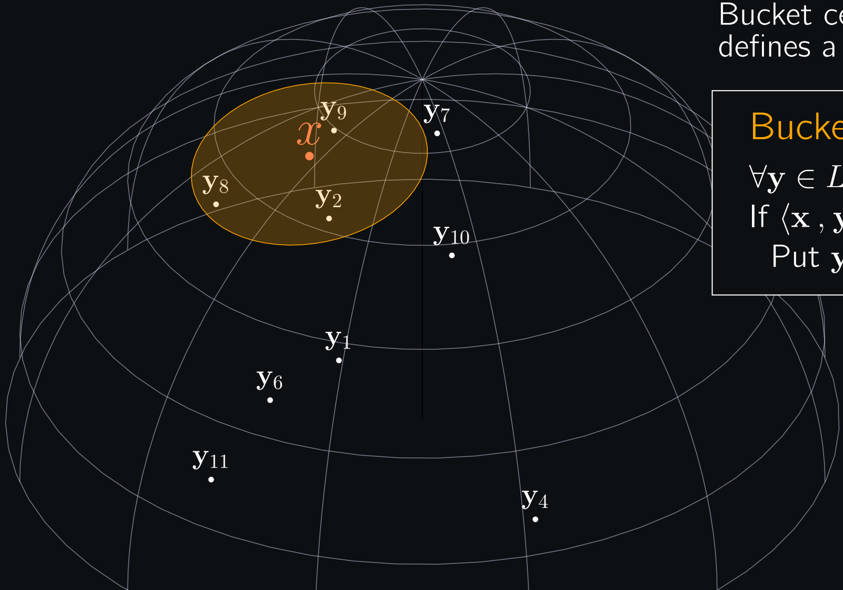
$$\Pr_{\substack{\mathbf{x},\mathbf{y}\sim\mathcal{S}^{d-1} \\ \mathrm{dist}(\mathbf{x},\mathbf{y})<r}}[h(\mathbf{x})=h(\mathbf{y})] \gg \Pr_{\mathbf{x},\mathbf{y}\sim\mathcal{S}^{d-1}}[h(\mathbf{x})=h(\mathbf{y})]$$

LSH on the unit sphere



Bucket center $\mathbf{x} \in \mathcal{S}^{d-1}$
defines a region $\mathsf{B}_{\mathbf{x}}$

LSH on the unit sphere



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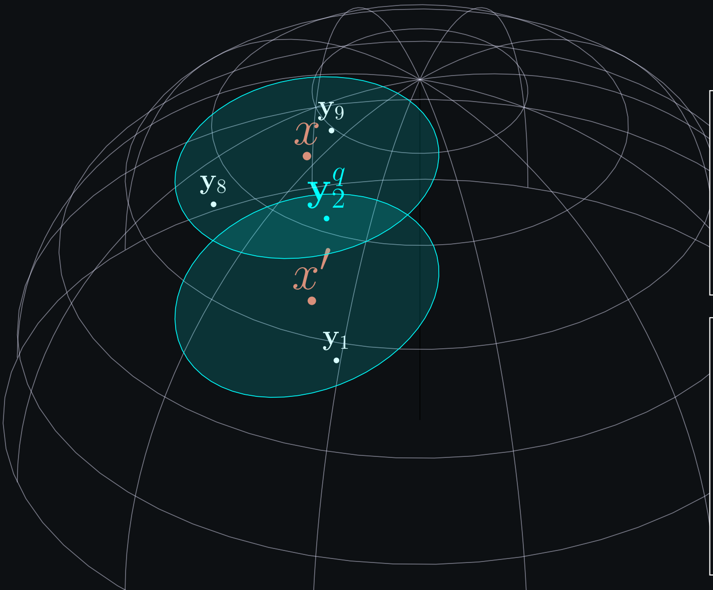
Bucketing phase

$\forall \mathbf{y} \in L$:

If $\langle \mathbf{x}\,, \mathbf{y} \rangle \geq \alpha$:

Put $\mathbf{y}$ into $\mathtt{B}_{\mathbf{x}}$

LSH on the unit sphere



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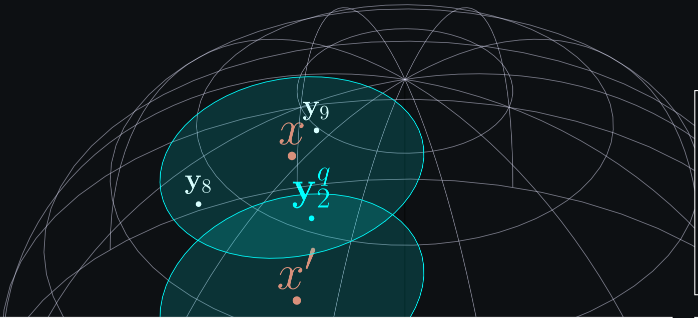
```

Query phase

```
$\forall \mathsf{B}_{\mathbf{x}}$ s.t. $\mathbf{y}^q \in \mathsf{B}_{\mathbf{x}}$ :
  $\forall \mathbf{y} \in \mathsf{B}_{\mathbf{x}}$ :
    Find $\mathbf{y}' \in \mathsf{B}_{\mathbf{x}}$ s.t.
    dist$(\mathbf{y}', \mathbf{y}^q) < r$
```

LSH on the unit sphere



Bucket center $\mathbf{x} \in \mathcal{S}^{d-1}$ defines a region $\mathsf{B}_{\mathbf{x}}$

Bucketing phase

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∀$\mathbf{y} \in L$ :
If $\langle \mathbf{x}, \mathbf{y} \rangle \geq \alpha$ :
  Put $\mathbf{y}$ into $\mathsf{B}_{\mathbf{x}}$
```

The number of $\mathbf{x}$'s determines the runtime

The shape of $\mathbf{x}$'s determines the complexity of finding $\mathbf{x}$ for $\mathbf{y}$.

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Our results (informal)

Instantiating LSH with Spherical caps, i.e.,

$$B_{\mathbf{x}}(\alpha) := \{\mathbf{y} \in \mathcal{S}^{d-1} \,:\, \langle \mathbf{x} \,, \mathbf{y} \rangle \leq \alpha\},$$

is optimal in the Euclidean metric and almost optimal in the Hamming metric.

Here optimal means that choosing hash regions different from spherical caps will not asymptotically improve the performance of LSH.

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Consequences:

- Another hashing strategy will not improve the performance of lattice sieving
- Improving only the closest pair subroutine in ISD will not result in a noticeable gain
- Asymtotically fastest algorithm will choose $\mathbf{x}$'s from a fast-decodable spherical code (may not exist for arbitrary dimensions).

Proof technique: Euclidean metric

Convolution on $\mathcal{S}^{d-1}$:

$$\mathcal{T}(f,g,h) := \int\int_{\mathcal{S}^{d-1}\times\mathcal{S}^{d-1}} f(\mathbf{x})g(\mathbf{y})h(\langle\mathbf{x}\ ,\mathbf{y}\rangle)\mathrm{d}\sigma(\mathbf{x})\mathrm{d}\sigma(\mathbf{y}).$$

$f,g:\mathcal{S}^{d-1}\to\mathbb{R}$, $h:[-1,1]\to\mathbb{R}$, σ – normalized surface measure on $\mathcal{S}^{d-1}$.

For which f,g,h is $\mathcal{T}(f,g,h)$ maximized?

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For which f, g, h is $\mathcal{T}(f, g, h)$ maximized?

Baernstein–Taylor rearrangement inequality on $\mathcal{S}^{d-1}$ [BT76] :

$$\mathcal{T}(f, g, h) \leq \mathcal{T}(f^\star, g^\star, h),$$

for $f^\star, g^\star$ depending only on $\mathbf{x}_1$ and is non-decreasing in $\mathbf{x}_1$,
$\sigma(\{f^\star > \lambda\}) = \sigma(\{f > \lambda\})$, $\sigma(\{g^\star > \lambda\}) = \sigma(\{g > \lambda\}), \forall\lambda.$

Proof technique: Euclidean metric

Take

$U, Q \subset \mathcal{S}^{d-1}$ – arbitrary sets

$$C_Q = \{\mathbf{z} \in \mathcal{S}^{d-1} : \mathbf{z}_1 \geq \alpha\}$$
$$C_U = \{\mathbf{z} \in \mathcal{S}^{d-1} : \mathbf{z}_1 \geq \alpha\}$$

$$\sigma(U) = \sigma(C_U), \sigma(Q) = \sigma(C_Q)$$

$f = \mathbf{1}(U)$

$g = \mathbf{1}(Q)$

$h(s) = \mathbf{1}\{s > r\}, r \in [-1, 1]$

$f^\star = \mathbf{1}(C_U)$

$g^\star = \mathbf{1}(C_Q)$

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$$C_Q = \{\mathbf{z} \in \mathcal{S}^{d-1} : \mathbf{z}_1 \geq \alpha\}$$
$$C_U = \{\mathbf{z} \in \mathcal{S}^{d-1} : \mathbf{z}_1 \geq \alpha\}$$

$\sigma(U) = \sigma(C_U), \sigma(Q) = \sigma(C_Q)$

$f^\star = \mathbf{1}(C_U)$

$g^\star = \mathbf{1}(C_Q)$

Notice that $(f, g, h, f^\star, g^\star)$, satisfy the BT inequality.

Interpreting integrals as probabilities leads to:

$$\Pr_{\mathbf{x},\mathbf{y} \sim \mathcal{S}^{d-1}}[\mathbf{x} \in U, \mathbf{y} \in Q \mid \langle \mathbf{x}\,, \mathbf{y} \rangle \geq r] \leq \Pr_{\mathbf{x},\mathbf{y} \sim \mathcal{S}^{d-1}}[\mathbf{x} \in C_U, \mathbf{y} \in C_Q \,|\langle \mathbf{x}\,, \mathbf{y} \rangle \geq r].$$

Proof technique: Hamming metric

Andoni-Razenshteyn inequality [AR16]:

For every hash function $h : \{0,1\}^d \to \mathbb{Z}$ and every $0 < r \leq d/2$:

$$\Pr_{\substack{\mathbf{x},\mathbf{y}\sim\{0,1\}^d \\ \mathbf{E}(\mathsf{dist}(\mathbf{x},\mathbf{y}))=r}} [h(\mathbf{x}) = h(\mathbf{y})] \leq \Pr_{\mathbf{x},\mathbf{y}\sim\{0,1\}^d} [h(\mathbf{x}) = h(\mathbf{y})]^{\frac{r}{d-r}}.$$

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Andoni-Razenshteyn inequality [AR16]:

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$\implies$ the lower bound on the runtime T of the closest pairs problem over $\{0,1\}^d$:

$$\log_2 T \geq \frac{1}{1 - r/d} \log_2 |L| .$$

- [MO15] achieves the lower bound in the sparse setting,
- and comes close to it in the dense setting.

Proof technique: Hamming metric

Source of the gap: for an arbitrary set $A \subset \{0,1\}^d$:

$$\Pr_{\substack{\mathbf{x},\mathbf{y}\sim\{0,1\}^d \\ \mathbf{E}(\mathsf{dist}(\mathbf{x},\mathbf{y}))=r}} [\mathbf{x} \in A \,|\mathbf{y} \in A] \leq \left(\frac{|A|}{2^d}\right)^{\frac{r}{d-r}} .$$

We need A, for which the above is tight. For spherical caps in $\{0,1\}^d$ it is not.

Interpretation of the result

- The result does not imply a lower bound on all possible sieving algorithms. Another use of the closest pairs problem or a completely different technique is possible.
- E.g., there is no contradiction with the result “Lattice sieving via quantum random walks” (eprint 2021/570).
- It does imply that we have an optimal near neighbor subroutine within sieving algorithms.
- It does imply that in order to noticeably improve ISD, another technique is needed.

References

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