THEOREM 2-9. *The linear Diophantine equation* $ax + by = c$ *has a solution if and only if* $d \mid c$, *where* $d = \gcd(a, b)$. *If* x_0, y_0 *is any particular solution of this equation, then all other solutions are given by*

$$x = x_0 + (b/d)t, \qquad y = y_0 - (a/d)t$$

for varying integers t.

Proof: To establish the second assertion of the theorem, let us suppose that a solution x_0, y_0 of the given equation is known. If x', y' is any other solution, then

$$ax_0 + by_0 = c = ax' + by',$$

which is equivalent to

$$a(x' - x_0) = b(y_0 - y').$$

By the Corollary to Theorem 2-4, there exist relatively prime integers r and s such that $a = dr$, $b = ds$. Substituting these values into the last-written equation and cancelling the common factor d, we find that

$$r(x' - x_0) = s(y_0 - y').$$

The situation is now this: $r \mid s(y_0 - y')$, with $\gcd(r, s) = 1$. Using Euclid's Lemma, it must be the case that $r \mid (y_0 - y')$; or, in other words, $y_0 - y' = rt$ for some integer t. Substituting, we obtain

$$x' - x_0 = st.$$

This leads us to the formulas

$$x' = x_0 + st = x_0 + (b/d)t,$$
$$y' = y_0 - rt = y_0 - (a/d)t.$$

It is easy to see that these values satisfy the Diophantine equation, regardless of the choice of the integer t; for,

$$\begin{aligned} ax' + by' &= a[x_0 + (b/d)t] + b[y_0 - (a/d)t] \\ &= (ax_0 + by_0) + (ab/d - ab/d)t \\ &= c + 0 \cdot t = c. \end{aligned}$$

Thus there are an infinite number of solutions of the given equation, one for each value of t.

Example 2-3

Consider the linear Diophantine equation

$$172x + 20y = 1000.$$

Applying Euclid's Algorithm to the evaluation of gcd (172, 20), we find that

$$\begin{aligned}
172 &= 8 \cdot 20 + 12,\\
20 &= 1 \cdot 12 + 8,\\
12 &= 1 \cdot 8 + 4,\\
8 &= 2 \cdot 4,
\end{aligned}$$

whence gcd (172, 20) = 4. Since $4 \mid 1000$, a solution to this equation exists. To obtain the integer 4 as a linear combination of 172 and 20, we work backwards through the above calculations, as follows:

$$\begin{aligned}
4 &= 12 - 8\\
&= 12 - (20 - 12)\\
&= 2 \cdot 12 - 20\\
&= 2(172 - 8 \cdot 20) - 20\\
&= 2 \cdot 172 + (-17)20.
\end{aligned}$$

Upon multiplying this relation by 250, one arrives at

$$\begin{aligned}
1000 = 250 \cdot 4 &= 250[2 \cdot 172 + (-17)20]\\
&= 500 \cdot 172 + (-4250)20,
\end{aligned}$$

so that $x = 500$ and $y = -4250$ provides one solution to the Diophantine equation in question. All other solutions are expressed by

$$\begin{aligned}
x &= 500 + (20/4)t = 500 + 5t,\\
y &= -4250 - (172/4)t = -4250 - 43t
\end{aligned}$$

for some integer t.

A little further effort produces the solutions in the positive integers, if any happen to exist. For this, t must be chosen so as to satisfy simultaneously the inequalities

$$5t + 500 > 0, \qquad -43t - 4250 > 0$$

or, what amounts to the same thing,

$$-98\tfrac{36}{43} > t > -100.$$

Since t must be an integer, we are forced to conclude that $t = -99$. Thus our Diophantine equation has a unique positive solution $x = 5$, $y = 7$ corresponding to the value $t = -99$.

It might be helpful to record the form that Theorem 2-9 takes when the coefficients are relatively prime integers.

COROLLARY. *If* $\gcd(a, b) = 1$ *and if* x_0, y_0 *is a particular solution of the linear Diophantine equation* $ax + by = c$, *then all solutions are given by*

$$x = x_0 + bt, \qquad y = y_0 - at$$

for integral values of t.

For example: The equation $5x + 22y = 18$ has $x_0 = 8, y_0 = -1$ as one solution; from the Corollary, a complete solution is given by $x = 8 + 22t, y = -1 - 5t$ for arbitrary t.

Diophantine equations frequently arise in the solving of certain types of traditional "word problems," as evidenced by our next example.

Example 2-4

A customer bought a dozen pieces of fruit, apples and oranges, for \$1.32. If an apple costs 3 cents more than an orange and more apples than oranges were purchased, how many pieces of each kind were bought?

To set up this problem as a Diophantine equation, let x be the number of apples and y the number of oranges purchased; also, let z represent the cost (in cents) of an orange. Then the conditions of the problem lead to

$$(z + 3)x + zy = 132$$

or equivalently

$$3x + (x + y)z = 132.$$

Since $x + y = 12$, the above equation may be replaced by

$$3x + 12z = 132,$$

which in turn simplifies to $x + 4z = 44$.

Stripped of inessentials, the object is to find integers x and z satisfying the Diophantine equation

$$(*) \qquad x + 4z = 44.$$

Inasmuch as gcd $(1, 4) = 1$ is a divisor of 44, there is a solution to this equation. Upon multiplying the relation $1 = 1(-3) + 4 \cdot 1$ by 44 to get

$$44 = 1(-132) + 4 \cdot 44,$$

it follows that $x_0 = -132$, $z_0 = 44$ serves as one solution. All other solutions of $(*)$ are of the form

$$x = -132 + 4t,$$
$$z = 44 - t,$$

where t is an integer.

Not all of the infinite set of values of t furnish solutions to the original problem. Only values of t should be considered which will ensure that $12 \geq x > 6$. This requires obtaining those t such that

$$12 \geq -132 + 4t > 6.$$

Now, $12 \geq -132 + 4t$ implies that $t \leq 36$, while $-132 + 4t > 6$ gives $t > 34\frac{1}{2}$. The only integral values of t to satisfy both inequalities are $t = 35$ and $t = 36$. Thus there are two possible purchases: a dozen apples costing 11 cents apiece (the case where $t = 36$), or else 8 apples at 12 cents each and 4 oranges at 9 cents each (the case where $t = 35$).

PROBLEMS 2.4

1. Determine all solutions in the integers of each of the following Diophantine equations:

   (a) $56x + 72y = 40$;
   (b) $24x + 138y = 18$;
   (c) $221x + 91y = 117$;
   (d) $84x - 438y = 156$.

2. Determine all solutions in the positive integers of each of the following Diophantine equations:

   (a) $30x + 17y = 300$;
   (b) $54x + 21y = 906$;
   (c) $123x + 360y = 99$;
   (d) $158x - 57y = 7$.

3. If a and b are relatively prime positive integers, prove that the Diophantine equation $ax - by = c$ has infinitely many solutions in the positive integers.

[*Hint:* There exist integers x_0 and y_0 such that $ax_0 + by_0 = 1$. For any integer t, which is larger than both $|x_0|/b$ and $|y_0|/a$, $x = x_0 + bt$ and $y = -(y_0 - at)$ are a positive solution of the given equation.]

4. (a) Prove that the Diophantine equation $ax + by + cz = d$ is solvable in the integers if and only if gcd (a, b, c) divides d.
   (b) Find all solutions in the integers of $15x + 12y + 30z = 24$. [*Hint:* Put $y = 3s - 5t$ and $z = -s + 2t$.]
5. (a) A man has \$4.55 in change composed entirely of dimes and quarters. What are the maximum and minimum number of coins that he can have? Is it possible for the number of dimes to equal the number of quarters?
   (b) The neighborhood theater charges \$1.80 for adult admissions and 75 cents for children. On a particular evening the total receipts were \$90. Assuming that more adults than children were present, how many people attended?
   (c) A certain number of sixes and nines are added to give a sum of 126; if the number of sixes and nines are interchanged, the new sum is 114. How many of each were there originally?
6. A farmer purchased one hundred head of livestock for a total cost of \$4000. Prices were as follows: calves, \$120 each; lambs, \$50 each; piglets, \$25 each. If the farmer obtained at least one animal of each type how many did he buy?
7. When Mr. Smith cashed a check at his bank, the teller mistook the number of cents for the number of dollars and vice versa. Unaware of this, Mr. Smith spent 68 cents and then noticed to his surprise that he had twice the amount of the original check. Determine the smallest value for which the check could have been written. [*Hint:* If x is the number of dollars and y the number of cents in the check, then $100y + x - 68 = 2(100x + y)$.]