primes, respectively, are 41, 47, 53 and 251, 257, 263, 269. Not long ago, a computer search revealed progressions of five and six consecutive primes, the terms having a common difference of 30; these begin with the primes

$$9{,}843{,}019 \qquad \text{and} \qquad 121{,}174{,}811.$$

We are not able to discover, at least for the time being, an arithmetic progression consisting of seven consecutive primes. When the restriction that the prime numbers involved be consecutive is removed, then it is possible to find infinitely many sets of seven primes in an arithmetic progression; one such is 7, 157, 307, 457, 607, 757, 907.

In interests of completeness, we might mention another famous problem that so far has resisted the most determined attack. For centuries, mathematicians have sought a simple formula that would yield every prime number or, failing this, a formula that would produce nothing but primes. At first glance, the request seems modest enough: find a function $f(n)$ whose domain is, say, the nonnegative integers and whose range is some infinite subset of the set of all primes. It was widely believed in the Middle Ages that the quadratic polynomial

$$f(n) = n^2 + n + 41$$

assumed only prime values. As evidenced by the following table, the claim is a correct one for $n = 0, 1, 2, \ldots, 39$.

n	$f(n)$	n	$f(n)$	n	$f(n)$
0	41	14	251	28	853
1	43	15	281	29	911
2	47	16	313	30	971
3	53	17	347	31	1033
4	61	18	383	32	1097
5	71	19	421	33	1163
6	83	20	461	34	1231
7	97	21	503	35	1301
8	113	22	547	36	1373
9	131	23	593	37	1447
10	151	24	641	38	1523
11	173	25	691	39	1601
12	197	26	743		
13	223	27	797		

However, this provocative conjecture is shattered in the cases $n = 40$ and $n = 41$, where there is a factor of 41:

$$f(40) = 40 \cdot 41 + 41 = 41^2$$

and

$$f(41) = 41 \cdot 42 + 41 = 41 \cdot 43.$$

The next value $f(42) = 1747$ turns out to be prime once again. It is not presently known whether $f(n) = n^2 + n + 41$ assumes infinitely many prime values for integral n.

The failure of the above function to be prime-producing is no accident, for it is easy to prove that there is no nonconstant polynomial $f(n)$ with integral coefficients which takes on just prime values for integral n. We assume that such a polynomial $f(n)$ actually does exist and argue until a contradiction is reached. Let

$$f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0,$$

where the coefficients $a_0, a_1, \ldots, a_k$ are all integers and $a_k \neq 0$. For a fixed value of n, say $n = n_0$, $p = f(n_0)$ is a prime number. Now, for any integer t, we consider the expression $f(n_0 + tp)$:

$$\begin{aligned} f(n_0 + tp) &= a_k(n_0 + tp)^k + \cdots + a_1(n_0 + tp) + a_0 \\ &= (a_k n_0{}^k + \cdots + a_1 n_0 + a_0) + pQ(t) \\ &= f(n_0) + pQ(t) \\ &= p + pQ(t) = p(1 + Q(t)), \end{aligned}$$

where $Q(t)$ is a polynomial in t having integral coefficients. Our reasoning shows that $p \mid f(n_0 + tp)$; hence, from our own assumption that $f(n)$ takes on only prime values, $f(n_0 + tp) = p$ for any integer t. Since a polynomial of degree k cannot assume the same value more than k times, we have obtained the required contradiction.

Recent years have seen a measure of success in the search for prime-producing functions. W. H. Mills proved (1947) that there exists a positive real number r such that the expression $f(n) = [r^{3^n}]$ is prime for $n = 1, 2, 3, \ldots$ (the bracket indicates the greatest integer function). Needless to say, this is strictly an existence theorem and nothing is known about the actual value of r.

PROBLEMS 3.3

1. Verify that the integers 1949 and 1951 are twin primes.
2. (a) If 1 is added to a product of twin primes, prove that a perfect square is always obtained.
   (b) Show that the sum of twin primes p and $p + 2$ is divisible by 12, provided that $p > 3$.
3. Find all pairs of primes p and q satisfying $p - q = 3$.
4. Sylvester (1896) rephrased Goldbach's Conjecture so as to read: Every even integer $2n$ greater than 4 is the sum of two primes, one larger than $n/2$ and the other less than $3n/2$. Verify this version of the conjecture for all even integers between 6 and 76.
5. In 1752, Goldbach submitted the following conjecture to Euler: Every odd integer can be written in the form $p + 2a^2$, where p is either a prime or 1 and $a \geq 0$. Show that the integer 5777 refutes this conjecture.
6. Prove that Goldbach's Conjecture that every even integer greater than 2 is the sum of two primes is equivalent to the statement that every integer greater than 5 is the sum of three primes. [*Hint:* If $2n - 2 = p_1 + p_2$, then $2n = p_1 + p_2 + 2$ and $2n + 1 = p_1 + p_2 + 3$.]
7. A conjecture of Lagrange (1775) asserts that every odd integer greater than 5 can be written as a sum $p_1 + 2p_2$, where p_1, p_2 are both primes. Confirm this for all odd integers through 75.
8. Given a positive integer n, it can be shown that there exists an even integer a which is representable as the sum of two odd primes in n different ways. Confirm that the integers 60, 78, and 84 can be written as the sum of two primes in six, seven, and eight ways, respectively.
9. (a) For $n > 3$, show that the integers n, $n + 2$, $n + 4$ cannot all be prime.
   (b) Three integers p, $p + 2$, $p + 6$ which are all prime are called a *prime-triplet.* Find five sets of prime-triplets.
10. Establish that the sequence

$$(n+1)! - 2, \quad (n+1)! - 3, \ldots, (n+1)! - (n+1)$$

produces n consecutive composite integers.

11. Find the smallest positive integer n for which the function $f(n) = n^2 + n + 17$ is composite. Do the same for the functions $g(n) = n^2 + 21n + 1$ and $h(n) = 3n^2 + 3n + 23$.
12. The following result was conjectured by Bertrand, but first proved by Tchebychef in 1850: For every positive integer $n > 1$, there exists at least one prime p satisfying $n < p < 2n$. Use Bertrand's Conjecture to show that $p_n < 2^n$, where p_n is the nth prime.
13. Apply the same method of proof as in Theorem 3-6 to show that there are infinitely many primes of the form $6n + 5$.

14. Find a prime divisor of the integer $N = 4(3 \cdot 7 \cdot 11) - 1$ of the form $4n + 3$. Do the same for $N = 4(3 \cdot 7 \cdot 11 \cdot 15) - 1$.

15. Another unanswered question is whether there exist an infinite number of sets of five consecutive odd integers of which four are primes. Find five such sets of integers.

16. Let the sequence of primes, with 1 adjoined, be denoted by $p_0 = 1, p_1 = 2, p_2 = 3, p_3 = 5, \ldots$. For each $n \geq 1$, it is known that there exists a suitable choice of coefficients $\varepsilon_k = \pm 1$ such that

$$p_{2n} = p_{2n-1} + \sum_{k=0}^{2n-2} \varepsilon_k p_k, \quad p_{2n+1} = 2p_{2n} + \sum_{k=0}^{2n-1} \varepsilon_k p_k.$$

To illustrate:

$$13 = 1 + 2 - 3 - 5 + 7 + 11 \quad \text{and} \quad 17 = 1 + 2 - 3 - 5 + 7 - 11 + 2 \cdot 13.$$

Determine similar representations for the primes 23, 29, 31, and 37.

17. In 1848 de Polignac claimed that every odd integer is the sum of a prime and a power of 2. For example, $55 = 47 + 2^3 = 23 + 2^5$. Show that the integers 509 and 877 discredit this claim.

18. (a) If p is a prime and $p \nmid b$, prove that in the arithmetic progression

$$a, a + b, a + 2b, a + 3b, \ldots$$

every pth term is divisible by p. [*Hint:* Since $\gcd(p, b) = 1$, there exists integers r and s satisfying $pr + bs = 1$. Put $n_k = kp - as$ for $k = 1, 2, \ldots$ and show that $p \mid (a + n_k b)$.]

(b) From part (a), conclude that if b is an odd integer, then every other term in the indicated progression is even.

19. In 1950, it was proven that any integer $n > 9$ can be written as a sum of distinct odd primes. Express the integers 25, 69, 81, and 125 in this fashion.

20. If p and $p^2 + 8$ are both prime numbers, prove that $p^3 + 4$ is also prime.

21. (a) For any integer $k > 0$, establish that the arithmetic progression

$$a + b, a + 2b, a + 3b, \ldots,$$

where $\gcd(a, b) = 1$, contains k consecutive terms which are composite. [*Hint:* Put $n = (a + b)(a + 2b) \cdots (a + kb)$ and consider the k terms

$$a + (n + 1)b, a + (n + 2)b, \ldots, a + (n + k)b.]$$

(b) Find five consecutive composite terms in the arithmetic progression

$$6, 11, 16, 21, 26, 31, 36, \ldots.$$

22. Show that 13 is the largest prime that can divide two successive integers of the form $n^2 + 3$.

23. (a) The arithmetic mean of the twin primes 5 and 7 is the triangular number 6. Are there any other twin primes with triangular mean?

(b) The arithmetic mean of the twin primes 3 and 5 is the perfect square 4. Are there any other twin primes with a square mean?

24. Determine all twin primes p and $q = p + 2$ for which $pq - 2$ is also prime.