6.1 THE FUNCTIONS τ AND σ

Certain functions are found to be of special importance in connection with the study of the divisors of an integer. Any function whose domain of definition is the set of positive integers is said to be a *number-theoretic (or arithmetic) function.* While the value of a number-theoretic function is not required to be a positive integer or, for that matter, even an integer, most of the number-theoretic functions that we shall encounter are integer-valued. Among the easiest to handle, as well as the most natural, are the functions τ and σ.

DEFINITION 6-1. Given a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and $\sigma(n)$ denote the sum of these divisors.

For an example of these notions, consider $n = 12$. Since 12 has the positive divisors 1, 2, 3, 4, 6, 12, we find that

$$\tau(12) = 6 \quad \text{and} \quad \sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28.$$

For the first few integers,

$$\tau(1) = 1,\ \tau(2) = 2,\ \tau(3) = 2,\ \tau(4) = 3,\ \tau(5) = 2,\ \tau(6) = 4,\ \ldots$$

and

$$\sigma(1) = 1,\ \sigma(2) = 3,\ \sigma(3) = 4,\ \sigma(4) = 7,\ \sigma(5) = 6,\ \sigma(6) = 12,\ \ldots.$$

It is not difficult to see that $\tau(n) = 2$ if and only if n is a prime number; also, $\sigma(n) = n + 1$ and if only if n is a prime.

Before studying the functions τ and σ in more detail, we wish to introduce a notation that will clarify a number of situations later on. It is customary to interpret the symbol

$$\sum_{d \mid n} f(d)$$

to mean, "Sum the values $f(d)$ as d runs over all the positive divisors of the positive integer n." For instance, we have

$$\sum_{d \mid 20} f(d) = f(1) + f(2) + f(4) + f(5) + f(10) + f(20).$$

With this understanding, τ and σ may be expressed in the form

$$\tau(n) = \sum_{d \mid n} 1, \quad \sigma(n) = \sum_{d \mid n} d.$$

The notation $\sum_{d \mid n} 1$, in particular, says that we are to add together as many 1's as there are positive divisors of n. To illustrate: the integer 10 has the four positive divisors 1, 2, 5, 10, whence

$$\tau(10) = \sum_{d \mid 10} 1 = 1 + 1 + 1 + 1 = 4,$$

while

$$\sigma(10) = \sum_{d \mid 10} d = 1 + 2 + 5 + 10 = 18.$$

Our first theorem makes it easy to obtain the positive divisors of a positive integer n once its prime factorization is known.

THEOREM 6-1. *If* $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ *is the prime factorization of* $n > 1$, *then the positive divisors of* n *are precisely those integers* d *of the form*

$$d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where $0 \leq a_i \leq k_i$ $(i = 1, 2, \ldots, r)$.

Proof: Note that the divisor $d = 1$ is obtained when $a_1 = a_2 = \cdots = a_r = 0$, and n itself occurs when $a_1 = k_1$, $a_2 = k_2$, $\ldots$, $a_r = k_r$. Suppose that d divides n nontrivially; say $n = dd'$, where $d > 1$, $d' > 1$. Express both d and d' as products of (not necessarily distinct) primes:

$$d = q_1 q_2 \cdots q_s, \qquad d' = t_1 t_2 \cdots t_u,$$

with q_i, t_j prime. Then

$$p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} = q_1 \cdots q_s t_1 \cdots t_u$$

are two prime factorizations of the positive integer n. By the uniqueness of the prime factorization, each prime q_i must be one of the p_j. Collecting the equal primes into a single integral power, we get

$$d = q_1 q_2 \cdots q_s = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where the possibility that $a_i = 0$ is allowed.

Conversely, every number $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ $(0 \leq a_i \leq k_i)$ turns out to be a divisor of n. For we can write

$$\begin{aligned} n &= p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \\ &= (p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r})(p_1^{k_1 - a_1} p_2^{k_2 - a_2} \cdots p_r^{k_r - a_r}) \\ &= dd', \end{aligned}$$

with $d' = p_1^{k_1 - a_1} p_2^{k_2 - a_2} \cdots p_r^{k_r - a_r}$ and $k_i - a_i \geq 0$ for each i. Then $d' > 0$ and $d \mid n$.

We put this theorem to work at once.

THEOREM 6-2. *If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then*

(a) $\tau(n) = (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$, *and*

(b) $\sigma(n) = \dfrac{p_1^{k_1+1} - 1}{p_1 - 1} \dfrac{p_2^{k_2+1} - 1}{p_2 - 1} \cdots \dfrac{p_r^{k_r+1} - 1}{p_r - 1}.$

Proof: According to Theorem 6-1, the positive divisors of n are precisely those integers

$$d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r},$$

where $0 \leq a_i \leq k_i$. There are $k_1 + 1$ choices for the exponent a_1; $k_2 + 1$ choices for $a_2, \ldots$; $k_r + 1$ choices for a_r; hence, there are

$$(k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$$

possible divisors of n.

In order to evaluate $\sigma(n)$, consider the product

$$(1 + p_1 + p_1^2 + \cdots + p_1^{k_1})(1 + p_2 + p_2^2 + \cdots + p_2^{k_2}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{k_r}).$$

Each positive divisor of n appears once and only once as a term in the expansion of this product, so that

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{k_1}) \cdots (1 + p_r + p_r^2 + \cdots + p_r^{k_r}).$$

Applying the formula for the sum of a finite geometric series to the ith factor on the right-hand side, we get

$$1 + p_i + p_i^2 + \cdots + p_i^{k_i} = \frac{p_i^{k_i+1} - 1}{p_i - 1}.$$

It follows that

$$\sigma(n) = \frac{p_1^{k_1+1} - 1}{p_1 - 1} \frac{p_2^{k_2+1} - 1}{p_2 - 1} \cdots \frac{p_r^{k_r+1} - 1}{p_r - 1}.$$

Corresponding to the $\sum$ notation for sums, a notation for products may be defined using the Greek capital letter "pi." The restriction delimiting the numbers over which the product is to be made is usually put under the $\prod$-sign. Examples are

$$\prod_{1 \le d \le 5} f(d) = f(1)f(2)f(3)f(4)f(5),$$

$$\prod_{d \mid 9} f(d) = f(1)f(3)f(9),$$

$$\prod_{\substack{p \mid 30 \\ p \text{ prime}}} f(d) = f(2)f(3)f(5).$$

With this convention, the conclusion to Theorem 6-2 takes the compact form: if $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

$$\tau(n) = \prod_{1 \le i \le r} (k_i + 1)$$

and

$$\sigma(n) = \prod_{1 \le i \le r} \frac{p_i^{k_i+1} - 1}{p_i - 1}$$

Example 6-1

The number $180 = 2^2 \cdot 3^2 \cdot 5$ has

$$\tau(180) = (2+1)(2+1)(1+1) = 18$$

positive divisors. These are integers of the form

$$2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3},$$

where $a_1 = 0, 1, 2$; $a_2 = 0, 1, 2$; $a_3 = 0, 1$. Specifically, we obtain

1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 30, 36, 45, 60, 90, 180.