The sum of these integers is

$$\sigma(180) = \frac{2^3-1}{2-1}\,\frac{3^3-1}{3-1}\,\frac{5^2-1}{5-1} = \frac{7}{1}\,\frac{26}{2}\,\frac{24}{4} = 7\cdot 13\cdot 6 = 546.$$

One of the more interesting properties of the divisor function τ is that the product of the positive divisors of an integer $n > 1$ is equal to $n^{\tau(n)/2}$. It is not difficult to get at this fact: Let d denote an arbitrary positive divisor of n, so that $n = dd'$ for some d'. As d ranges over all $\tau(d)$ positive divisors of n, $\tau(d)$ such equations occur. Multiplying these together, we get

$$n^{\tau(n)} = \prod_{d\mid n} d \cdot \prod_{d'\mid n} d'.$$

But as d runs through the divisors of n, so does d'; hence, $\prod_{d\mid n} d = \prod_{d'\mid n} d'$. The situation is now this:

$$n^{\tau(n)} = \left(\prod_{d\mid n} d\right)^2$$

or equivalently,

$$n^{\tau(n)/2} = \prod_{d\mid n} d.$$

The reader might (or, at any rate, should) have one lingering doubt concerning this equation. For it is by no means obvious that the left-hand side is always an integer. If $\tau(n)$ is even, there is certainly no problem. When $\tau(n)$ is odd, n turns out to be a perfect square (Problem 7), say $n = m^2$; thus $n^{\tau(n)/2} = m^{\tau(n)}$, settling all suspicions.

For a numerical example, the product of the five divisors of 16 (namely, 1, 2, 4, 8, 16) is

$$\prod_{d\mid 16} d = 16^{\tau(16)/2} = 16^{5/2} = 4^5 = 1024.$$

Multiplicative functions arise naturally in the study of the prime factorization of an integer. Before presenting the definition, we observe that

$$\tau(2\cdot 10) = \tau(20) = 6 \neq 2\cdot 4 = \tau(2)\cdot\tau(10).$$

At the same time

$$\sigma(2\cdot 10) = \sigma(20) = 42 \neq 3\cdot 18 = \sigma(2)\cdot\sigma(10).$$

These calculations bring out the nasty fact that, in general, it need not be true that

$$\tau(mn) = \tau(m)\tau(n) \quad \text{and} \quad \sigma(mn) = \sigma(m)\sigma(n).$$

On the positive side of the ledger, equality always holds provided we stick to relatively prime m and n. This circumstance is what prompts

DEFINITION 6-2. A number-theoretic function f is said to be *multiplicative* if

$$f(mn) = f(m)f(n)$$

whenever $\gcd(m, n) = 1$.

For simple illustrations of multiplicative functions, one need only consider the functions given by $f(n) = 1$ and $g(n) = n$ for all $n \geq 1$. It follows by induction that if f is multiplicative and $n_1, n_2, \ldots, n_r$ are positive integers which are pairwise relatively prime, then

$$f(n_1 n_2 \cdots n_r) = f(n_1)f(n_2) \cdots f(n_r).$$

Multiplicative functions have one big advantage for us: they are completely determined once their values at prime powers are known. Indeed, if $n > 1$ is a given positive integer, then we can write $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ in canonical form; since the $p_i^{k_i}$ are relatively prime in pairs, the multiplicative property ensures that

$$f(n) = f(p_1^{k_1})f(p_2^{k_2}) \cdots f(p_r^{k_r}).$$

If f is a multiplicative function which does not vanish identically, then there exists an integer n such that $f(n) \neq 0$. But

$$f(n) = f(n \cdot 1) = f(n)f(1).$$

Being nonzero, $f(n)$ may be cancelled from both sides of this equation to give $f(1) = 1$. The point to which we wish to call attention is that $f(1) = 1$ for any multiplicative function not identically zero.

We now establish that τ and σ have the multiplicative property.

THEOREM 6-3. *The functions τ and σ are both multiplicative functions.*

Proof: Let m and n be relatively prime integers. Since the result is trivially true if either m or n is equal to 1, we may assume that $m > 1$ and $n > 1$. If

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r} \quad \text{and} \quad n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$$

are the prime factorizations of m and n, then, since $\gcd(m, n) = 1$, no p_i can occur among the q_j. It follows that the prime factorization of the product mn is given by

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}.$$

Appealing to Theorem 6-2, we obtain

$$\begin{aligned}\tau(mn) &= [(k_1 + 1) \cdots (k_r + 1)][(j_1 + 1) \cdots (j_s + 1)] \\ &= \tau(m)\tau(n).\end{aligned}$$

In a similar fashion, Theorem 6-2 gives

$$\begin{aligned}\sigma(mn) &= \left[\frac{p_1^{k_1+1} - 1}{p_1 - 1} \cdots \frac{p_r^{k_r+1} - 1}{p_r - 1}\right]\left[\frac{q_1^{j_1+1} - 1}{q_1 - 1} \cdots \frac{q_s^{j_s+1} - 1}{q_s - 1}\right] \\ &= \sigma(m)\sigma(n).\end{aligned}$$

Thus, τ and σ are multiplicative functions.

We continue our program by proving a general result on multiplicative functions. This requires a preparatory lemma.

LEMMA. *If* $\gcd(m, n) = 1$, *then the set of positive divisors of* mn *consists of all products* $d_1 d_2$, *where* $d_1 \mid n$, $d_2 \mid m$ *and* $\gcd(d_1, d_2) = 1$; *furthermore, these products are all distinct.*

Proof: It is harmless to assume that $m > 1$ and $n > 1$; let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $n = q_1^{j_1} q_2^{j_2} \cdots q_s^{j_s}$ be their respective prime factorizations. Inasmuch as the primes $p_1, \ldots, p_r, q_1, \ldots, q_s$ are all distinct, the prime factorization of mn is

$$mn = p_1^{k_1} \cdots p_r^{k_r} q_1^{j_1} \cdots q_s^{j_s}.$$

Hence, any positive divisor d of mn will be uniquely representable in the form

$$d = p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s}, \qquad 0 \leq a_i \leq k_i,\ 0 \leq b_i \leq j_i.$$

This allows us to write d as $d = d_1 d_2$, where $d_1 = p_1^{a_1} \cdots p_r^{a_r}$ divides m and $d_2 = q_1^{b_1} \cdots q_s^{b_s}$ divides n. Since no p_i is equal to any q_j, we surely have $\gcd(d_1, d_2) = 1$.

A keystone in much of our subsequent work is

THEOREM 6-4. *If f is a multiplicative function and F is defined by*

$$F(n) = \sum_{d \mid n} f(d),$$

then F is also multiplicative.

Proof: Let m and n be relatively prime positive integers. Then

$$F(mn) = \sum_{d \mid mn} f(d) = \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1 d_2),$$

since every divisor d of mn can be uniquely written as a product of a divisor d_1 of m and a divisor d_2 of n, where $\gcd(d_1, d_2) = 1$. By the definition of a multiplicative function,

$$f(d_1 d_2) = f(d_1) f(d_2).$$

It follows that

$$\begin{aligned} F(mn) &= \sum_{\substack{d_1 \mid m \\ d_2 \mid n}} f(d_1) f(d_2) \\ &= \left(\sum_{d_1 \mid m} f(d_1) \right) \left(\sum_{d_2 \mid n} f(d_2) \right) = F(m) F(n). \end{aligned}$$

It might be helpful to take time out and run through the proof of Theorem 6-4 in a concrete case. Letting $m = 8$ and $n = 3$, we have

$$\begin{aligned} F(8 \cdot 3) &= \sum_{d \mid 24} f(d) \\ &= f(1) + f(2) + f(3) + f(4) + f(6) + f(8) + f(12) + f(24) \\ &= f(1 \cdot 1) + f(2 \cdot 1) + f(1 \cdot 3) + f(4 \cdot 1) + f(2 \cdot 3) + f(8 \cdot 1) \\ &\qquad + f(4 \cdot 3) + f(8 \cdot 3) \\ &= f(1)f(1) + f(2)f(1) + f(1)f(3) + f(4)f(1) + f(2)f(3) + f(8)f(1) \\ &\qquad + f(4)f(3) + f(8)f(3) \\ &= [f(1) + f(2) + f(4) + f(8)][f(1) + f(3)] \\ &= \sum_{d \mid 8} f(d) \cdot \sum_{d \mid 3} f(d) = F(8)F(3). \end{aligned}$$

Theorem 6-4 provides a deceptively short way of drawing the conclusion that τ and σ are multiplicative.