COROLLARY. *The functions τ and σ are multiplicative functions.*

Proof: We have mentioned before that the constant function $f(n) = 1$ is multiplicative, as is the identity function $f(n) = n$. Since τ and σ may be represented in the form

$$\tau(n) = \sum_{d\mid n} 1 \qquad \text{and} \qquad \sigma(n) = \sum_{d\mid n} d,$$

the stated result follows immediately from Theorem 6-4.

PROBLEMS 6.1

1. Let m and n be positive integers and $p_1, p_2, \ldots, p_r$ be the distinct primes which divide at least one of m or n. Then m and n may be written in the form

$$m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}, \qquad \text{with } k_i \geq 0 \text{ for } i = 1, 2, \ldots, r$$
$$n = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r}, \qquad \text{with } j_i \geq 0 \text{ for } i = 1, 2, \ldots, r$$

Prove that

$$\gcd(m, n) = p_1^{u_1} p_2^{u_2} \cdots p_r^{u_r}, \quad \operatorname{lcm}(m, n) = p_1^{v_1} p_2^{v_2} \cdots p_r^{v_r},$$

where $u_i = \min\{k_i, j_i\}$, the smaller of k_i and j_i; and $v_i = \max\{k_i, j_i\}$, the larger of k_i and j_i.

2. Use Problem 1 to calculate gcd (12378, 3054) and lcm (12378, 3054).
3. Deduce from Problem 1 that $\gcd(m, n) \operatorname{lcm}(m, n) = mn$ for positive integers m and n.
4. In the notation of Problem 1, show that $\gcd(m, n) = 1$ if and only if $k_i j_i = 0$ for $i = 1, 2, \ldots, r$.
5. (a) Verify that $\tau(n) = \tau(n+1) = \tau(n+2) = \tau(n+3)$ holds for $n = 3655$ and 4503.
   (b) When $n = 14$, 206, and 957, show that $\sigma(n) = \sigma(n+1)$.
6. For any integer $n \geq 1$, establish the inequality $\tau(n) \leq 2\sqrt{n}$. [*Hint:* If $d \mid n$, then one of d or n/d is less than or equal to $\sqrt{n}$.]
7. Prove that:
   (a) $\tau(n)$ is an odd integer if and only if n is a perfect square;
   (b) $\sigma(n)$ is an odd integer if and only if n is a perfect square or twice a perfect square. [*Hint:* If p is an odd prime, then $1 + p + p^2 + \cdots + p^k$ is odd only when k is even.]
8. Show that $\sum_{d\mid n} 1/d = \sigma(n)/n$ for every positive integer n.

9. If n is a square-free integer, prove that $\tau(n) = 2^r$, where r is the number of prime divisors of n.

10. Establish the assertions below:
 (a) If $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ is the prime factorization of $n > 1$, then

$$1 > \frac{n}{\sigma(n)} > \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right).$$

 (b) For any positive integer n, $\sigma(n!)/n! \geq 1 + 1/2 + 1/3 + \cdots + 1/n$. [*Hint:* See Problem 8.]
 (c) If $n > 1$ is a composite number, then $\sigma(n) > n + \sqrt{n}$. [*Hint:* Let $d \mid n$, where $1 < d < n$, so $1 < n/d < n$. If $d \leq \sqrt{n}$, then $n/d \geq \sqrt{n}$.]

11. Given a positive integer $k > 1$, show that there are infinitely many integers n for which $\tau(n) = k$, but at most finitely many n with $\sigma(n) = k$. [*Hint:* Utilize Problem 10(a).]

12. (a) Find the form of all positive integers n satisfying $\tau(n) = 10$. What is the smallest positive integer for which this is true?
 (b) Show that there are no positive integers n satisfying $\sigma(n) = 10$. [*Hint:* Note that for $n > 1$, $\sigma(n) > n$.]

13. Prove that there are infinitely many pairs of integers m and n with $\sigma(m^2) = \sigma(n^2)$. [*Hint:* Choose k such that $\gcd(k, 10) = 1$ and consider the integers $m = 5k$, $n = 4k$.]

14. For $k \geq 2$, show each of the following:
 (a) $n = 2^{k-1}$ satisfies the equation $\sigma(n) = 2n - 1$;
 (b) if $2^k - 1$ is prime, then $n = 2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$;
 (c) if $2^k - 3$ is prime, then $n = 2^{k-1}(2^k - 3)$ satisfies the equation $\sigma(n) = 2n + 2$.

 It is not known if there are any integers n for which $\sigma(n) = 2n + 1$.

15. If n and $n + 2$ are twin primes, establish that $\sigma(n + 2) = \sigma(n) + 2$; this also holds for $n = 434$ and 8575.

16. (a) For any integer $n > 1$, prove that there exist integers n_1 and n_2 with $\tau(n_1) + \tau(n_2) = n$.
 (b) Prove that Goldbach's Conjecture implies that for each even integer $2n$ there exist integers n_1 and n_2 with $\sigma(n_1) + \sigma(n_2) = 2n$.

17. For a fixed integer k, show that the function f defined by $f(n) = n^k$ is multiplicative.

18. Let f and g be multiplicative functions such that $f(p^k) = g(p^k)$ for each prime p and $k \geq 1$. Prove that $f = g$.

19. Prove that if f and g are multiplicative functions, then so is their product fg and quotient f/g (whenever the latter function is defined).

20. Define the function ρ by taking $\rho(1)=1$ and $\rho(n)=2^r$, if the prime factorization of $n>1$ is $n=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$. For instance, $\rho(8)=2$ and $\rho(10)=\rho(36)=2^2$.
(a) Deduce that ρ is a multiplicative function.
(b) Find a formula for $F(n)=\sum_{d\mid n}\rho(d)$ in terms of the prime factorization of n.

21. For any positive integer n, prove that $\sum_{d\mid n}\tau(d)^3=(\sum_{d\mid n}\tau(d))^2$. [*Hint:* Both sides of the equation in question are multiplicative functions of n, so that it suffices to consider the case $n=p^k$, where p is a prime.]

22. Given $n\geq 0$, let $\sigma_s(n)$ denote the sum of the sth powers of the positive divisors of n; that is,

$$\sigma_s(n)=\sum_{d\mid n}d^s.$$

Verify the following:
(a) $\sigma_0=\tau$ and $\sigma_1=\sigma$.
(b) σ_s is a multiplicative function. [*Hint:* The function f, defined by $f(n)=n^s$, is multiplicative.]
(c) If $n=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ is the prime factorization of n, then

$$\sigma_s(n)=\left(\frac{p_1^{s(k_1+1)}-1}{p_1^s-1}\right)\left(\frac{p_2^{s(k_2+1)}-1}{p_2^s-1}\right)\cdots\left(\frac{p_r^{s(k_r+1)}-1}{p_r^s-1}\right).$$

23. For any positive integer n, show that

(a) $\displaystyle\sum_{d\mid n}\sigma(d)=\sum_{d\mid n}\frac{n}{d}\tau(d)$, and

(b) $\displaystyle\sum_{d\mid n}\frac{n}{d}\sigma(d)=\sum_{d\mid n}d\tau(d)$

[*Hint:* Since the functions $F(n)=\sum_{d\mid n}\sigma(d)$ and $G(n)=\sum_{d\mid n}n/d\;\tau(d)$ are both multiplicative, it suffices to prove that $F(p^k)=G(p^k)$ for any prime p.]

6.2 THE MÖBIUS INVERSION FORMULA

We introduce another naturally defined function on the positive integers, the Möbius μ-function.

Definition 6-3. For a positive integer n, define μ by the rules

$$\mu(n)=\begin{cases}1 \text{ if } n=1\\ 0 \text{ if } p^2\mid n \text{ for some prime } p\\ (-1)^r \text{ if } n=p_1p_2\cdots p_r, \text{ where the } p_i \text{ are distinct primes}\end{cases}$$

Put somewhat differently, Definition 6-3 states that $\mu(n)=0$ if n is not a square-free integer, while $\mu(n)=(-1)^r$ if n is square-free with r prime factors. For example: $\mu(30)=\mu(2\cdot 3\cdot 5)=(-1)^3=-1$. The first few values of μ are

$$\mu(1)=1,\ \mu(2)=-1,\ \mu(3)=-1,\ \mu(4)=0,\ \mu(5)=-1,\ \mu(6)=1,\ \ldots .$$

If p is a prime number, it is clear that $\mu(p)=-1$; also, $\mu(p^k)=0$ for $k\geq 2$.

As the reader may have guessed already, the Möbius μ-function is multiplicative. This is the content of

THEOREM 6-5. *The function μ is a multiplicative function.*

Proof: We want to show that $\mu(mn)=\mu(m)\mu(n)$, whenever m and n are relatively prime. If either $p^2 \mid m$ or $p^2 \mid n$, p a prime, then $p^2 \mid mn$; hence, $\mu(mn)=0=\mu(m)\mu(n)$, and the formula holds trivially. We may therefore assume that both m and n are square-free integers. Say, $m=p_1 p_2\cdots p_r$, $n=q_1 q_2\cdots q_s$, the primes p_i and q_j being all distinct. Then

$$\begin{aligned}\mu(mn)&=\mu(p_1\cdots p_r q_1\cdots q_s)=(-1)^{r+s}\\ &=(-1)^r(-1)^s=\mu(m)\mu(n),\end{aligned}$$

which completes the proof.

Let us see what happens if $\mu(d)$ is evaluated for all the positive divisors d of an integer n and the results added. In case $n=1$, the answer is easy; here,

$$\sum_{d\mid 1}\mu(d)=\mu(1)=1.$$

Suppose that $n>1$ and put

$$F(n)=\sum_{d\mid n}\mu(d).$$

To prepare the ground, we first calculate $F(n)$ for the power of a prime, say, $n=p^k$. The positive divisors of p^k are just the $k+1$ integers $1, p, p^2, \ldots, p^k$, so that

$$\begin{aligned}F(p^k)&=\sum_{d\mid p^k}\mu(d)=\mu(1)+\mu(p)+\mu(p^2)+\cdots+\mu(p^k)\\ &=\mu(1)+\mu(p)=1+(-1)=0.\end{aligned}$$