Since μ is known to be a multiplicative function, an appeal to Theorem 6-4 is legitimate; this result guarantees that F is multiplicative too. Thus, if the canonical factorization of n is $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, then $F(n)$ is the product of the values assigned to F for the prime powers in this representation:

$$F(n) = F(p_1^{k_1})F(p_2^{k_2}) \cdots F(p_r^{k_r}) = 0.$$

We record this result as

THEOREM 6-6. *For each positive integer $n \geq 1$,*

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

where d runs through the positive divisors of n.

For an illustration of this last theorem, consider $n = 10$. The divisors of 10 are 1, 2, 5, 10 and the desired sum is

$$\sum_{d|10} \mu(d) = \mu(1) + \mu(2) + \mu(5) + \mu(10)$$
$$= 1 + (-1) + (-1) + 1 = 0.$$

The full significance of Möbius' function should become apparent with the next theorem.

THEOREM 6-7 (Möbius Inversion Formula). *Let F and f be two number-theoretic functions related by the formula*

$$F(n) = \sum_{d|n} f(d).$$

Then

$$f(n) = \sum_{d|n} \mu(d)F(n/d) = \sum_{d|n} \mu(n/d)F(d).$$

Proof: The two sums mentioned in the conclusion of the theorem are seen to be the same upon replacing the dummy index d by $d' = n/d$; as d ranges over all positive divisors of n, so does d'.

Carrying out the required computation, we get

$$(1) \quad \sum_{d|n} \mu(d)F(n/d) = \sum_{d|n} \left(\mu(d) \sum_{c|(n/d)} f(c) \right) = \sum_{d|n} \left(\sum_{c|(n/d)} \mu(d)f(c) \right).$$

It is easily verified that $d \mid n$ and $c \mid (n/d)$ if and only if $c \mid n$ and $d \mid (n/c)$. Because of this, the last expression in (1) becomes

$$\sum_{d \mid n} \Big(\sum_{c \mid (n/d)} \mu(d) f(c) \Big) = \sum_{c \mid n} \Big(\sum_{d \mid (n/c)} f(c) \mu(d) \Big) \tag{2}$$

$$= \sum_{c \mid n} \Big(f(c) \sum_{d \mid (n/c)} \mu(d) \Big).$$

In compliance with Theorem 6-6, the sum $\sum_{d \mid (n/c)} \mu(d)$ must vanish except when $n/c = 1$ (that is, when $n = c$), in which case it is equal to 1; the upshot is that the right-hand side of (2) simplifies to

$$\sum_{c \mid n} \Big(f(c) \sum_{d \mid (n/c)} \mu(d) \Big) = \sum_{c = n} f(c) \cdot 1 = f(n),$$

giving us the stated result.

Let us use $n = 10$ again to illustrate how the double sum in (2) is turned around. In this instance, we find that

$$\begin{aligned}\sum_{d \mid 10} \Big(\sum_{c \mid (10/d)} \mu(d) f(c) \Big) &= \mu(1)[f(1) + f(2) + f(5) + f(10)] \\ &\quad + \mu(2)[f(1) + f(5)] + \mu(5)[f(1) + f(2)] + \mu(10) f(1) \\ &= f(1)[\mu(1) + \mu(2) + \mu(5) + \mu(10)] \\ &\quad + f(2)[\mu(1) + \mu(5)] + f(5)[\mu(1) + \mu(2)] + f(10)\mu(1) \\ &= \sum_{c \mid 10} \Big(\sum_{d \mid (10/c)} f(c) \mu(d) \Big).\end{aligned}$$

To see how Möbius inversion works in a particular case, we remind the reader that the functions τ and σ may both be described as "sum functions":

$$\tau(n) = \sum_{d \mid n} 1 \quad \text{and} \quad \sigma(n) = \sum_{d \mid n} d.$$

Theorem 6-7 tells us that these formulas may be inverted to give

$$1 = \sum_{d \mid n} \mu(n/d) \tau(d) \quad \text{and} \quad n = \sum_{d \mid n} \mu(n/d) \sigma(d),$$

valid for all $n \geq 1$.

Theorem 6-4 insures that if f is a multiplicative function, then so is $F(n) = \sum_{d \mid n} f(d)$. Turning the situation around, one might ask whether the multiplicative nature of F forces that of f. Surprisingly enough, this is exactly what happens.

THEOREM 6-8. *If F is a multiplicative function and*

$$F(n) = \sum_{d|n} f(d),$$

then f is also multiplicative.

Proof: Let m and n be relatively prime positive integers. We recall that any divisor d of mn can be uniquely written as $d = d_1 d_2$, where $d_1 \mid m$, $d_2 \mid n$, and $\gcd(d_1, d_2) = 1$. Thus, using the inversion formula,

$$\begin{aligned} f(mn) &= \sum_{d|mn} \mu(d) F\left(\frac{mn}{d}\right) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1 d_2) F\left(\frac{mn}{d_1 d_2}\right) \\ &= \sum_{\substack{d_1|m \\ d_2|n}} \mu(d_1)\mu(d_2) F\left(\frac{m}{d_1}\right) F\left(\frac{n}{d_2}\right) \\ &= \sum_{d_1|m} \mu(d_1) F\left(\frac{m}{d_1}\right) \sum_{d_2|n} \mu(d_2) F\left(\frac{n}{d_2}\right) = f(m)f(n), \end{aligned}$$

which is the assertion of the theorem. Needless to say, the multiplicative character of μ and of F is crucial to the above calculation.

PROBLEMS 6.2

1. (a) For each positive integer n, show that

$$\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0.$$

   (b) For any integer $n \geq 3$, show that $\sum_{k=1}^{n} \mu(k!) = 1$.

2. The *Mangoldt function* Λ is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \text{ where } p \text{ is a prime and } k \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

   Prove that $\Lambda(n) = \sum_{d|n} \mu(n/d) \log d = -\sum_{d|n} \mu(d) \log d$. [*Hint:* First show that $\sum_{d|n} \Lambda(d) = \log n$ and then apply the Möbius Inversion Formula.]

3. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ be the prime factorization of the integer $n > 1$. If f is a multiplicative function, prove that

$$\sum_{d \mid n} \mu(d) f(d) = (1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_r)).$$

[*Hint:* By Theorem 6-4, the function F defined by $F(n) = \sum_{d \mid n} \mu(d) f(d)$ is multiplicative; hence, $F(n)$ is the product of the values $F(p_i^{k_i})$.]

4. If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, use Problem 3 to establish the following:

(a) $\sum_{d \mid n} \mu(d) \tau(d) = (-1)^r$;

(b) $\sum_{d \mid n} \mu(d) \sigma(d) = (-1)^r p_1 p_2 \cdots p_r$;

(c) $\sum_{d \mid n} \mu(d)/d = (1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_r)$;

(d) $\sum_{d \mid n} d\mu(d) = (1 - p_1)(1 - p_2) \cdots (1 - p_r)$.

5. Let $S(n)$ denote the number of square-free divisors of n. Establish that

$$S(n) = \sum_{d \mid n} |\mu(d)| = 2^r$$

where r is the number of distinct prime divisors of n. [*Hint:* S is a multiplicative function.]

6. Find formulas for $\sum_{d \mid n} \mu^2(d)/\tau(d)$ and $\sum_{d \mid n} \mu^2(d)/\sigma(d)$ in terms of the prime factorization of n.

7. The *Liouville λ-function* is defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^{k_1 + k_2 + \cdots + k_r}$, if the prime factorization of $n > 1$ is $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. For instance, $\lambda(360) = \lambda(2^3 \cdot 3^2 \cdot 5) = (-1)^{3+2+1} = (-1)^6 = 1$.

(a) Prove that λ is a multiplicative function.

(b) Given a positive integer n, verify that

$$\sum_{d \mid n} \lambda(d) = \begin{cases} 1 & \text{if } n = m^2 \text{ for some integer } m \\ 0 & \text{otherwise} \end{cases}$$

8. If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, establish that $\sum_{d \mid n} \mu(d) \lambda(d) = 2^r$.