Proof: Noting that $\tau(n) = \sum_{d|n} 1$, we may write τ for F and take f to be the constant function $f(n) = 1$ for all n.

In the same way, the relation $\sigma(n) = \sum_{d|n} d$ yields

COROLLARY 2. *If N is a positive integer, then*

$$\sum_{n=1}^{N} \sigma(n) = \sum_{n=1}^{N} n[N/n].$$

These last two corollaries are perhaps clarified with an example.

Example 6-3

Consider the case $N = 6$. The results on page 110 tell us that

$$\sum_{n=1}^{6} \tau(n) = 14.$$

From Corollary 1,

$$\begin{aligned}\sum_{n=1}^{6} [6/n] &= [6] + [3] + [2] + [3/2] + [6/5] + [1] \\ &= 6 + 3 + 2 + 1 + 1 + 1 = 14,\end{aligned}$$

as it should. In the present case, we also have

$$\sum_{n=1}^{6} \sigma(n) = 33,$$

while a simple calculation leads to

$$\begin{aligned}\sum_{n=1}^{6} n[6/n] &= 1[6] + 2[3] + 3[2] + 4[3/2] + 5[6/5] + 6[1] \\ &= 1 \cdot 6 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 = 33.\end{aligned}$$

PROBLEMS 6.3

1. Given integers a and $b > 0$, show that there exists a unique integer r with $0 \leq r < b$ satisfying $a = [a/b]b + r$.
2. Let x and y be real numbers. Prove that the greatest integer function satisfies the following properties:
   (a) $[x + n] = [x] + n$ for any integer n.

(b) $[x]+[-x]=0$ or -1, according as x is an integer or not. [*Hint:* Write $x=[x]+\theta$, with $0\leq\theta<1$, so $-x=-[x]-1+(1-\theta)$.]
(c) $[x]+[y]\leq[x+y]$ and, when x and y are positive, $[x][y]\leq[xy]$.
(d) $[x/n]=[[x]/n]$ for any positive integer n. [*Hint:* Let $x/n=[x/n]+\theta$, where $0\leq\theta<1$; then $[x]=n[x/n]+[n\theta]$.]
(e) $[nm/k]\geq n[m/k]$ for positive integers n, m, k.
(f) $[x]+[y]+[x+y]\leq[2x]+[2y]$. [*Hint:* Let $x=[x]+\theta$, $0\leq\theta<1$, and $y=[y]+\theta'$, $0\leq\theta'<1$. Consider cases in which neither, one, or both of θ and θ' are greater than $\frac{1}{2}$.]

3. Find the highest power of 5 dividing 1000! and the highest power of 7 dividing 2000!.

4. Find the exponent of the highest power of the prime p dividing
(a) the product $2\cdot4\cdot6\cdots(2n)$ of the first n even integers;
(b) the product $1\cdot3\cdot5\cdots(2n-1)$ of the first n odd integers. [*Hint:* Note that $1\cdot3\cdot5\cdots(2n-1)=(2n)!/2^n n!$.]

5. Show that 1000! terminates in 249 zeroes.

6. If $n\geq1$ and p is a prime, prove that
(a) $(2n)!/(n!)^2$ is an even integer. [*Hint:* Use induction on n.]
(b) The exponent of the highest power of p which divides $(2n)!/(n!)^2$ is

$$\sum_{k=1}^{\infty}([2n/p^k]-2[n/p^k]).$$

(c) In the prime factorization of $(2n)!/(n!)^2$ the exponent of any prime p such that $n<p<2n$ is equal to 1.

7. Let the positive integer n be written in terms of powers of the prime p so that $n=a_kp^k+\cdots+a_2p^2+a_1p+a_0$, where $0\leq a_i<p$. Show that the exponent of the highest power of p appearing in the prime factorization of $n!$ is

$$\frac{n-(a_k+\cdots+a_2+a_1+a_0)}{p-1}.$$

8. (a) Using Problem 7, show that the exponent of highest power of p dividing $(p^k-1)!$ is $[p^k-(p-1)k-1]/(p-1)$. [*Hint:* Recall the identity $p^k-1=(p-1)(p^{k-1}+\cdots+p^2+p+1)$.]
(b) Determine the highest power of 3 dividing 80! and the highest power of 7 dividing 2400!. [*Hint:* $2400=7^4-1$.]

9. Find an integer $n\geq1$ such that the highest power of 5 contained in $n!$ is 100. [*Hint:* Since the sum of coefficients of the powers of 5 needed to express n in the base 5 is at least 1, begin by considering the equation $(n-1)/4=100$.]

10. Given a positive integer N, show that

(a) $\sum_{n=1}^{N} \mu(n)[N/n] = 1;$

(b) $\left|\sum_{n=1}^{N} \mu(n)/n\right| \leq 1.$

11. Illustrate Problem 10 in the case $N = 6$.

12. Verify that the formula

$$\sum_{n=1}^{N} \lambda(n)[N/n] = [\sqrt{N}]$$

holds for any positive integer N. [*Hint:* Apply Theorem 6-11 to the multiplicative function $F(n) = \sum_{d \mid n} \lambda(d)$, noting that there are $[\sqrt{n}]$ perfect squares not exceeding n.]