In this setting, we find that the desired sum is

$$1+7+11+13+17+19+23+29=120=\tfrac{1}{2}\cdot 30\cdot 8.$$

This is a good point at which to give an application of the Möbius Inversion Formula.

THEOREM 7-8. *For any positive integer $n$,*

$$\phi(n)=n\sum_{d\mid n}\mu(d)/d.$$

*Proof:* The proof is deceptively simple: If one applies the inversion formula to

$$F(n)=n=\sum_{d\mid n}\phi(d),$$

the result is

$$\phi(n)=\sum_{d\mid n}\mu(d)F(n/d)=\sum_{d\mid n}\mu(d)n/d.$$

Let us illustrate the situation with $n=10$ again. As can easily be seen,

$$\begin{aligned}10\sum_{d\mid 10}\mu(d)/d&=10[\mu(1)+\mu(2)/2+\mu(5)/5+\mu(10)/10]\\&=10[1+(-1)/2+(-1)/5+(-1)^2/10]\\&=10[1-1/2-1/5+1/10]=10\cdot 2/5=4=\phi(10).\end{aligned}$$

Starting with Theorem 7-8, it is an easy matter to determine the value of the phi-function for any positive integer $n$. Suppose that the prime-power decomposition of $n$ is $n=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ and consider the product

$$P=\prod_{p_i\mid n}(\mu(1)+\mu(p_i)/p_i+\cdots+\mu(p_i^{k_i})/p_i^{k_i}).$$

Multiplying this out, we obtain a sum of terms of the form

$$\mu(1)\mu(p_1^{a_1})\mu(p_2^{a_2})\cdots\mu(p_r^{a_r})/p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r},\qquad 0\le a_i\le k_i$$

or, since $\mu$ is known to be multiplicative,

$$\mu(p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r})/p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}=\mu(d)/d,$$

where the summation is over the set of divisors $d = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ of $n$. Hence, $P = \sum_{d \mid n} \mu(d)/d$. It follows from Theorem 7-8 that

$$\phi(n) = n \sum_{d \mid n} \mu(d)/d = n \prod_{p_i \mid n} (\mu(1) + \mu(p_i)/p_i + \cdots + \mu(p_i^{k_i})/p_i^{k_i}).$$

But $\mu(p_i^{a_i}) = 0$ whenever $a_i \geq 2$. As a result, the last-written equation reduces to

$$\phi(n) = n \prod_{p_i \mid n} (\mu(1) + \mu(p_i)/p_i) = n \prod_{p_i \mid n} (1 - 1/p_i),$$

which agrees with the formula established earlier by different reasoning. What is significant about this argument is that no assumption is made concerning the multiplicative character of the phi-function, only of $\mu$.

## PROBLEMS 7.4

1. For a positive integer $n$, prove that

$$\sum_{d \mid n} (-1)^{n/d} \phi(d) = \begin{cases} 0 \text{ if } n \text{ is even} \\ -n \text{ if } n \text{ is odd} \end{cases}$$

[*Hint*: If $n = 2^k N$, where $N$ is odd, then $\sum_{d \mid n} (-1)^{n/d} \phi(d) = \sum_{d \mid 2^{k-1}N} \phi(d) - \sum_{d \mid N} \phi(2^k d)$.]

2. Confirm that $\sum_{d \mid 36} \phi(d) = 36$ and $\sum_{d \mid 36} (-1)^{36/d} \phi(d) = 0$.

3. For a positive integer $n$, prove that $\sum_{d \mid n} \mu^2(d)/\phi(d) = n/\phi(n)$. [*Hint:* See the hint in Problem 1.]

4. Use Problem 3, Section 6.2, to give a different proof of the fact that

$$n \sum_{d \mid n} \mu(d)/d = \phi(n).$$

5. If the integer $n > 1$ has the prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, establish the following:

(a) $\displaystyle\sum_{d \mid n} \mu(d)\phi(d) = (2 - p_1)(2 - p_2) \cdots (2 - p_r)$

(b) $\displaystyle\sum_{d \mid n} d\phi(d) = \left(\frac{p_1^{2k_1+1} + 1}{p_1 + 1}\right)\left(\frac{p_2^{2k_2+1} + 1}{p_2 + 1}\right) \cdots \left(\frac{p_r^{2k_r+1} + 1}{p_r + 1}\right)$

(c) $\displaystyle\sum_{d \mid n} \phi(d)/d = \left(1 + \frac{k_1(p_1 - 1)}{p_1}\right)\left(1 + \frac{k_2(p_2 - 1)}{p_2}\right) \cdots \left(1 + \frac{k_r(p_r - 1)}{p_r}\right)$

[*Hint:* For part (a), use Problem 3, Section 6-2.]

6. Verify the formula $\sum_{d=1}^{n} \phi(d)[n/d] = n(n+1)/2$ for any positive integer $n$. [*Hint:* This is a direct application of Theorems 6-11 and 7-6.]
7. If $n$ is a square-free integer, prove that $\sum_{d \mid n} \sigma(d^{k-1})\phi(d) = n^k$ for all integers $k \geq 2$.
8. For a square-free integer $n > 1$, show that $\tau(n^2) = n$ if and only if $n = 3$.
9. Prove that $3 \mid \sigma(3n+2)$ and $4 \mid \sigma(4n+3)$ for any positive integer $n$.
10. (a) Given $k > 0$, establish that there exists a sequence of $k$ consecutive integers $n+1, n+2, \ldots, n+k$ satisfying
$$\mu(n+1) = \mu(n+2) = \cdots = \mu(n+k) = 0.$$
[*Hint:* Consider the system of linear congruences
$$x \equiv -1 \pmod 4,\ x \equiv -2 \pmod 9,\ \ldots,\ x \equiv -k \pmod{p_k^2}$$
where $p_k$ is the $k$th prime.]
(b) Find four consecutive integers for which $\mu(n) = 0$.
11. Prove the statements below:
(a) An integer $n$ is prime if and only if $\sigma(n) + \phi(n) = n\tau(n)$. [*Hint:* First derive the relation $\sum_{d \mid n} \sigma(d)\phi(n/d) = n\tau(n)$.]
(b) An integer $n$ is prime if and only if $\phi(n) \mid n-1$ and $n+1 \mid \sigma(n)$. [*Hint:* See Problem 11(a), Section 7-2.]
12. Show that there exist infinitely many integers $n$ such that $\phi(n) = n/3$, but none for which $\phi(n) = n/4$.
13. For $n > 2$, establish the inequality $\phi(n^2) + \phi((n+1)^2) \leq 2n^2$.