where j is the largest integer less than or equal to $(n-1)/2$. Derive this result. [*Hint:* Argue by induction, using the relation $u_n = u_{n-1} + u_{n-2}$; note also that $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$.]

12. Establish that for $n \geq 1$,

(a) $\binom{n}{1}u_1 + \binom{n}{2}u_2 + \binom{n}{3}u_3 + \cdots + \binom{n}{n}u_n = u_{2n}$;

(b) $-\binom{n}{1}u_1 + \binom{n}{2}u_2 - \binom{n}{3}u_3 + \cdots + (-1)^n\binom{n}{n}u_n = -u_n$.

13.3 FINITE CONTINUED FRACTIONS

In that part of the *Liber Abaci* dealing with the resolution of fractions into unit fractions, Fibonacci introduced a kind of "continued fraction." For example, he employed the symbol $\frac{1\ 1\ 1}{3\ 4\ 5}$ as an abbreviation for

$$\frac{1 + \frac{1 + \frac{1}{5}}{4}}{3} = \frac{1}{3} + \frac{1}{3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5}.$$

The modern practice is, however, to write continued fractions in a descending fashion, as with

$$2 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{3 + \frac{1}{2}}}}.$$

A multiple-decked expression of this type is said to be a finite simple continued fraction. To put the matter formally:

DEFINITION 13-1. By a *finite continued fraction* is meant a fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \; \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}},$$

where $a_0, a_1, \ldots, a_n$ are real numbers, all of which except possibly a_0 are positive. The numbers $a_1, a_2, \ldots, a_n$ are the *partial denominators* of this fraction. Such a fraction is called *simple* if all of the a_i are integers. integers.

While giving due credit to Fibonacci, most authorities agree that the theory of continued fractions begins with Rafael Bombelli, the last of the great algebraists of Renaissance Italy. In his *L'Algebra Opera* (1572), Bombelli attempted to find square roots by means of infinite continued fractions—a method both ingenious and novel. He essentially proved that $\sqrt{13}$ could be expressed as the continued fraction

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6 + \ddots}}}.$$

It may be interesting to mention that Bombelli was the first to popularize the work of Diophantus in the Latin West. He set out initially to translate the Vatican Library's copy of Diophantus' *Arithmetica* (probably the same manuscript uncovered by Regiomontanus), but, carried away by other labors, never finished the project. Instead he took all the problems of the first four Books and embodied them in his *Algebra*, interspersing them with his own problems. Although Bombelli did not distinguish between the problems, he nonetheless acknowledged that he had borrowed freely from the *Arithmetica*.

Evidently, the value of any finite simple continued fraction will always be a rational number. For instance, the continued fraction

$$3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{2}}}}$$

can be condensed to the value 170/53:

$$\begin{aligned} 3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{2}}}} &= 3 + \cfrac{1}{4 + \cfrac{1}{1 + \frac{2}{9}}} \\ &= 3 + \cfrac{1}{4 + \frac{9}{11}} \\ &= 3 + \tfrac{11}{53} \\ &= \tfrac{170}{53}. \end{aligned}$$

THEOREM 13-5. *Any rational number can be written as a finite simple continued fraction.*

Proof: Let a/b, where $b > 0$, be any rational number. Euclid's algorithm for finding the greatest common divisor of a and b gives us the equations

$$\begin{aligned}
a &= ba_0 + r_1, & 0 &< r_1 < b\\
b &= r_1 a_1 + r_2, & 0 &< r_2 < r_1\\
r_1 &= r_2 a_2 + r_3, & 0 &< r_3 < r_2\\
&\vdots\\
r_{n-2} &= r_{n-1} a_{n-1} + r_n, & 0 &< r_n < r_{n-1}\\
r_{n-1} &= r_n a_n + 0.
\end{aligned}$$

Notice that since each remainder r_k is a positive integer, $a_1, a_2, \ldots, a_n$ are all positive. Rewrite the equations of the algorithm in the following manner:

$$\begin{aligned}
a/b &= a_0 + r_1/b = a_0 + 1/(b/r_1),\\
b/r_1 &= a_1 + r_2/r_1 = a_1 + 1/(r_1/r_2),\\
r_1/r_2 &= a_2 + r_3/r_2 = a_2 + 1/(r_2/r_3),\\
&\vdots\\
r_{n-1}/r_n &= a_n.
\end{aligned}$$

If we eliminate b/r_1 from the first of these equations, then

$$a/b = a_0 + 1/(b/r_1) = a_0 + \cfrac{1}{a_1 + \cfrac{1}{(r_1/r_2)}}.$$

In this result, substitute the value of r_1/r_2 as given by the third equation:

$$a/b = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{(r_2/r_3)}}}.$$

Continuing in this way, we can go on to get

$$a/b = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{\ddots}{\qquad \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}},$$

thereby finishing the proof.

To illustrate the procedure involved in the proof of Theorem 13-5, let us represent 19/51 as a continued fraction. An application of Euclid's algorithm to the integers 19 and 51 gives the equations

$$\begin{array}{lcl} 51 = 2 \cdot 19 + 13 & \text{or} & 51/19 = 2 + 13/19, \\ 19 = 1 \cdot 13 + 6 & \text{or} & 19/13 = 1 + 6/13, \\ 13 = 2 \cdot 6 + 1 & \text{or} & 13/6 = 2 + 1/6, \\ 6 = 6 \cdot 1 + 0 & \text{or} & 6/6 = 1. \end{array}$$

Making the appropriate substitutions, it is seen that

$$\frac{19}{51} = \frac{1}{(51/19)} = \frac{1}{2 + \frac{13}{19}}$$

$$= \cfrac{1}{2 + \cfrac{1}{\frac{19}{13}}}$$

$$= \cfrac{1}{2 + \cfrac{1}{1 + \frac{6}{13}}}$$

$$= \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\frac{13}{6}}}}$$

$$= \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{6}}}}$$

which is the continued fraction expansion for 19/51.