Since continued fractions are unwieldy to print or write, we adopt the convention of denoting a continued fraction by a symbol which displays its partial quotients; say, by the symbol $[a_0; a_1, \ldots, a_n]$. In this notation, the expansion for 19/51 is indicated by

$$[0; 2, 1, 2, 6]$$

and for $172/51 = 3 + 19/51$ by

$$[3; 2, 1, 2, 6].$$

The initial integer in the symbol $[a_0; a_1, \ldots, a_n]$ will be zero when the value of the fraction is positive but less than one.

The representation of a rational number as a finite simple continued fraction is not unique: once the representation has been obtained, we can always modify the last term. For, if $a_n > 1$, then

$$a_n = (a_n - 1) + 1 = (a_n - 1) + \frac{1}{1},$$

where $a_n - 1$ is a positive integer, hence

$$[a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_n - 1, 1].$$

On the other hand, if $a_n = 1$, then

$$a_{n-1} + \frac{1}{a_n} = a_{n-1} + \frac{1}{1} = a_{n-1} + 1,$$

so that

$$[a_0; a_1, \ldots, a_{n-1}, a_n] = [a_0; a_1, \ldots, a_{n-2}, a_{n-1} + 1].$$

Every rational number has two representations as a simple continued fraction, one with an even number of partial denominators and one with an odd number (it turns out that these are the only two representations). In the case of 19/51,

$$19/51 = [0; 2, 1, 2, 6] = [0; 2, 1, 2, 5, 1].$$

Example 13-1

We go back to the Fibonacci sequence and consider the quotient of two successive Fibonacci numbers (that is, the rational number u_{n+1}/u_n) written as a simple continued fraction. As pointed out earlier, the Euclidean Algorithm for the greatest common divisor of u_n and u_{n+1} produces the $n-1$ equations

$$\begin{aligned} u_{n+1} &= 1 \cdot u_n + u_{n-1}, \\ u_n &= 1 \cdot u_{n-1} + u_{n-2}, \\ &\vdots \\ u_4 &= 1 \cdot u_3 + u_2, \\ u_3 &= 2 \cdot u_2 + 0. \end{aligned}$$

Since the quotients generated by the algorithm become the partial denominators of the continued fraction, we may write

$$u_{n+1}/u_n = [1; 1, 1, \ldots, 1, 2].$$

But u_{n+1}/u_n is also represented by a continued fraction having one more partial denominator than does $[1; 1, 1, \ldots, 1, 2]$; namely,

$$u_{n+1}/u_n = [1; 1, 1, \ldots, 1, 1, 1],$$

where the integer 1 appears $n+1$ times. Thus, the fraction u_{n+1}/u_n has a continued fraction expansion which is very easy to describe: there are n partial denominators all equal to 1.

As a final item on our program, we would like to indicate how the theory of continued fractions can be applied to the solution of linear Diophantine equations. This requires knowing a few pertinent facts about the "convergents" of a continued fraction, so let us begin proving them here.

DEFINITION 13-2. The continued fraction made from $[a_0; a_1, \ldots, a_n]$ by cutting off the expansion after the kth partial denominator a_k is called the kth *convergent* of the given continued fraction and denoted by C_k; in symbols,

$$C_k = [a_0; a_1, \ldots, a_k], \qquad (1 \leq k \leq n).$$

We let the zero'th convergent C_0 be equal to the number a_0.

A point worth calling attention to is that for $k < n$ if a_k is replaced by the value $a_k + 1/a_{k+1}$, then the convergent C_k becomes the convergent C_{k+1};

$$[a_0; a_1, \ldots, a_{k-1}, a_k + 1/a_{k+1}] = [a_0; a_1, \ldots, a_{k-1}, a_k, a_{k+1}] = C_{k+1}.$$

It hardly needs remarking that the last convergent C_n always equals the rational number represented by the original continued fraction.

Going back to our example $19/51 = [0; 2, 1, 2, 6]$, the successive convergents are

$$C_0 = 0,$$
$$C_1 = [0; 2] = 0 + \tfrac{1}{2} = \tfrac{1}{2},$$
$$C_2 = [0; 2, 1] = 0 + \frac{1}{2 + \frac{1}{1}} = \tfrac{1}{3},$$
$$C_3 = [0; 2, 1, 2] = 0 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{1}{2}}} = \tfrac{3}{8},$$
$$C_4 = [0; 2, 1, 2, 6] = 19/51.$$

Except for the last convergent C_4, these are alternately less than or greater than $19/51$, each convergent being closer to $19/51$ than the previous one.

Much of the labor in calculating the convergents of a continued fraction $[a_0; a_1, \ldots, a_n]$ can be avoided by establishing formulas for their numerators and denominators. To this end, let us define numbers p_k and q_k $(k = 0, 1, \ldots, n)$ as follows:

$$\begin{aligned} p_0 &= a_0 & q_0 &= 1 \\ p_1 &= a_1 a_0 + 1 & q_1 &= a_1 \\ p_k &= a_k p_{k-1} + p_{k-2} & q_k &= a_k q_{k-1} + q_{k-2} \end{aligned}$$

for $k = 2, 3, \ldots, n$.

A direct computation shows that the first few convergents of $[a_0; a_1, \ldots, a_n]$ are

$$C_0 = a_0 = \frac{a_0}{1} = \frac{p_0}{q_0},$$
$$C_1 = a_0 + \frac{1}{a_1} = \frac{a_1 a_0 + 1}{a_1} = \frac{p_1}{q_1},$$
$$C_2 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1} = \frac{p_2}{q_2}.$$

Success hinges on being able to show that this relationship continues to hold. This is the content of

THEOREM 13-6. *The kth convergent of the simple continued fraction* $[a_0; a_1, \ldots, a_n]$ *has the value*

$$C_k = p_k/q_k \qquad (0 \leq k \leq n).$$

Proof: The remarks above indicate that the theorem is true for $k = 0, 1, 2$. Let us assume that it is true for $k = m$, where $2 \leq m < n$; that is, for this m,

$$(*) \qquad C_m = p_m/q_m = \frac{a_m p_{m-1} + p_{m-2}}{a_m q_{m-1} + q_{m-2}}.$$

Note that the integers $p_{m-1}, q_{m-1}, p_{m-2}, q_{m-2}$ depend on the first $m - 1$ partial denominators $a_1, a_2, \ldots, a_{m-1}$, hence are independent of a_m. Thus formula $(*)$ remains valid if a_m is replaced by the value $a_m + 1/a_{m+1}$:

$$\left[a_0; a_1, \ldots, a_{m-1}, a_m + \frac{1}{a_{m+1}}\right] = \frac{\left(a_m + \dfrac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \dfrac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}}$$

As we have explained earlier, the effect of this substitution is to change C_m into the convergent C_{m+1}, so that

$$\begin{aligned} C_{m+1} &= \frac{\left(a_m + \dfrac{1}{a_{m+1}}\right) p_{m-1} + p_{m-2}}{\left(a_m + \dfrac{1}{a_{m+1}}\right) q_{m-1} + q_{m-2}} \\ &= \frac{a_{m+1}(a_m p_{m-1} + p_{m-2}) + p_{m-1}}{a_{m+1}(a_m q_{m-1} + q_{m-2}) + q_{m-1}} \\ &= \frac{a_{m+1} p_m + p_{m-1}}{a_{m+1} q_m + q_{m-1}}. \end{aligned}$$

But this is precisely the form the theorem should take in the case $k = m + 1$. So, by induction, the stated result holds.

Let us see how this works in a specific instance. In our example, $19/51 = [0; 2, 1, 2, 6]$:

$$\begin{aligned} p_0 &= 0 & &\text{and} & q_0 &= 1, \\ p_1 &= 0 \cdot 2 + 1 = 1 & & & q_1 &= 2, \\ p_2 &= 1 \cdot 1 + 0 = 1 & & & q_2 &= 1 \cdot 2 + 1 = 3, \\ p_3 &= 2 \cdot 1 + 1 = 3 & & & q_3 &= 2 \cdot 3 + 2 = 8, \\ p_4 &= 6 \cdot 3 + 1 = 19 & & & q_4 &= 6 \cdot 8 + 3 = 51. \end{aligned}$$