This says that the convergents of [0; 2, 1, 2, 6] are

$$C_0 = p_0/q_0 = 0,\ C_1 = p_1/q_1 = 1/2,\ C_2 = p_2/q_2 = 1/3,\ C_3 = p_3/q_3 = 3/8,$$
$$C_4 = p_4/q_4 = 19/51,$$

as we know that they should be.

We continue our development of the properties of convergents by proving

THEOREM 13-7. *If $C_k = p_k/q_k$ is the kth convergent of the simple continued fraction $[a_0; a_1, \ldots, a_n]$, then*

$$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}, \qquad 1 \leq k \leq n.$$

Proof: Induction on k works quite simply, with the relation

$$p_1 q_0 - q_1 p_0 = (a_1 a_0 + 1) \cdot 1 - a_1 \cdot a_0 = 1 = (-1)^{1-1},$$

disposing of the case $k = 1$. We assume that the formula in question is also true for $k = m$, where $1 \leq m < n$. Then

$$\begin{aligned} p_{m+1} q_m - q_{m+1} p_m &= (a_{m+1} p_m + p_{m-1}) q_m - (a_{m+1} q_m + q_{m-1}) p_m \\ &= -(p_m q_{m-1} - q_m p_{m-1}) \\ &= -(-1)^{m-1} = (-1)^m \end{aligned}$$

and so the formula holds for $m + 1$, whenever it holds for m. It follows by induction that it is valid for all k with $1 \leq k \leq n$.

A notable consequence of this result is that the numerator and denominator of any convergent are relatively prime, so that the convergents are always given in lowest terms.

COROLLARY. *For $1 \leq k \leq n$, p_k and q_k are relatively prime.*

Proof: If $d = \gcd(p_k, q_k)$, then from the theorem, $d \mid (-1)^{k-1}$; since $d > 0$, this forces us to conclude that $d = 1$.

Example 13-2

Consider the continued fraction [0; 1, 1, ..., 1] in which the partial denominators are all equal to 1. Here, the first few convergents are

$$C_0 = 0/1,\ C_1 = 1/1,\ C_2 = 2/1,\ C_3 = 3/2,\ C_4 = 5/3,\ \ldots.$$

Since the numerator of the kth convergent C_k is

$$p_k = 1 \cdot p_{k-1} + p_{k-2} = p_{k-1} + p_{k-2}$$

and the denominator is

$$q_k = 1 \cdot q_{k-1} + q_{k-2} = q_{k-1} + q_{k-2},$$

it is apparent that

$$C_k = u_{k+1}/u_k \qquad (k \geq 2),$$

where u_k denotes the kth Fibonacci number. In the present context, the identity $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$ of Theorem 13-7 assumes the form

$$u_{k+1} u_{k-1} - u_k^2 = (-1)^{k-1};$$

this is precisely formula (3) on page 294.

Let us now turn to the linear Diophantine equation

$$ax + by = c,$$

where a, b, c are given integers. Since no solution of this equation exists if $d \nmid c$, where $d = \gcd(a, b)$, there is no harm in assuming that $d \mid c$. In fact, we need only concern ourselves with the situation in which the coefficients are relatively prime. For if $\gcd(a, b) = d > 1$, then the equation may be divided by d to produce

$$(a/d)x + (b/d)y = c/d.$$

Both equations have the same solutions and, in the latter case, we know that $\gcd(a/d, b/d) = 1$.

Observe too that a solution of the equation

$$ax + by = c, \quad \gcd(a, b) = 1$$

may be obtained by first solving the Diophantine equation

$$ax + by = 1, \quad \gcd(a, b) = 1.$$

Indeed, if integers x_0 and y_0 can be found for which $ax_0 + by_0 = 1$, then multiplication of both sides by c gives

$$a(cx_0) + b(cy_0) = c.$$

Hence, $x = cx_0$ and $y = cy_0$ is the desired solution of $ax + by = c$.

To secure a pair of integers x and y satisfying the equation $ax + by = 1$, expand the rational number a/b as a simple continued fraction; say,

$$a/b = [a_0; a_1, \ldots, a_n].$$

Now the last two convergents of this continued fraction are

$$C_{n-1} = p_{n-1}/q_{n-1} \quad \text{and} \quad C_n = p_n/q_n = a/b.$$

Since gcd $(p_n, q_n) = 1 = \gcd(a, b)$, it may be concluded that

$$p_n = a \quad \text{and} \quad q_n = b.$$

By virtue of Theorem 13-7, we have

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$$

or, with a change of notation,

$$aq_{n-1} - bp_{n-1} = (-1)^{n-1}.$$

Thus, with $x = q_{n-1}$ and $y = -p_{n-1}$, we have

$$ax + by = (-1)^{n-1}.$$

If n is odd, the equation $ax + by = 1$ has the particular solution $x_0 = q_{n-1}, y_0 = -p_{n-1}$, while if n is an even integer, then a solution is given by $x_0 = -q_{n-1}, y_0 = p_{n-1}$. Our earlier theory tells us that the general solution is

$$x = x_0 + bt, y = y_0 - at, \qquad (t = 0, \pm 1, \pm 2, \ldots).$$

Example 13-3

Let us solve the linear Diophantine equation

$$172x + 20y = 1000$$

by means of simple continued fractions. Since gcd $(172, 20) = 4$, this equation may be replaced by the equation

$$43x + 5y = 250.$$

The first step is to find a particular solution to

$$43x + 5y = 1.$$

To accomplish this, we begin by writing 43/5 (or if one prefers, 5/43) as a simple continued fraction. The sequence of equalities obtained by applying the Euclidean Algorithm to the numbers 43 and 5 is

$$43 = 8 \cdot 5 + 3,$$
$$5 = 1 \cdot 3 + 2,$$
$$3 = 1 \cdot 2 + 1,$$
$$2 = 2 \cdot 1,$$

so that $43/5=[8;1,1,2]=8+\cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}}$. The convergents of this continued fraction are

$$C_0=8/1, C_1=9/1, C_2=17/2, C_3=43/5,$$

from which it follows that $p_2=17$, $q_2=2$, $p_3=43$ and $q_3=5$. Falling back on Theorem 13-7 again,

$$p_3q_2-q_3p_2=(-1)^{3-1},$$

or in equivalent terms,

$$43\cdot 2-5\cdot 17=1.$$

When this relation is multiplied by 250, we obtain

$$43\cdot 500+5(-4250)=250.$$

Thus a particular solution of the Diophantine equation $43x+5y=250$ is

$$x_0=500, y_0=-4250.$$

The general solution is given by the equations

$$x=500+5t, y=-4250-43t, \qquad (t=0, \pm 1, \pm 2, \ldots).$$

Before proving a theorem concerning the behavior of the odd and even numbered convergents of a simple continued fraction, a preliminary lemma is required.

LEMMA. *If q_k is the denominator of the kth convergent C_k of the simple continued fraction $[a_0; a_1, \ldots, a_n]$, then $q_{k-1}\leq q_k$ for $1\leq k\leq n$, with strict inequality when $k>1$.*

Proof: We establish the lemma by induction. In the first place, $q_0=1\leq a_1=q_1$, so that the asserted equality holds when $k=1$. Assume, then, that it is true for $k=m$, where $1\leq m<n$. Then

$$q_{m+1}=a_{m+1}q_m+q_{m-1}>a_{m+1}q_m\geq 1\cdot q_m=q_m$$

so that the inequality is also true for $k=m+1$.

With this information available, it is an easy matter to prove

THEOREM 13-8. (1) *The convergents with even subscripts form a strictly increasing sequence; that is,*

$$C_0<C_2<C_4<\cdots.$$