(2) *The convergents with odd subscripts form a strictly decreasing sequence; that is,*

$$C_1 > C_3 > C_5 > \cdots.$$

(3) *Every convergent with an odd subscript is greater than every convergent with an even subscript.*

Proof: With the aid of Theorem 13-7, we find that

$$\begin{aligned}
C_{k+2} - C_k &= (C_{k+2} - C_{k+1}) + (C_{k+1} - C_k) \\
&= \left(\frac{p_{k+2}}{q_{k+2}} - \frac{p_{k+1}}{q_{k+1}}\right) + \left(\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k}\right) \\
&= \frac{(-1)^{k+1}}{q_{k+2}q_{k+1}} + \frac{(-1)^k}{q_{k+1}q_k} \\
&= \frac{(-1)^k(q_{k+2} - q_k)}{q_k q_{k+1} q_{k+2}}.
\end{aligned}$$

Recalling that $q_i > 0$ for all $i \geq 0$ and that $q_{k+2} - q_k > 0$ by the lemma, it is evident that $C_{k+2} - C_k$ has the same algebraic sign as does $(-1)^k$. Thus, if k is an even integer, say $k = 2j$, then $C_{2j+2} > C_{2j}$; whence

$$C_0 < C_2 < C_4 < \cdots.$$

Similarly, if k is an odd integer, say $k = 2j - 1$, then $C_{2j+1} < C_{2j-1}$; whence

$$C_1 > C_3 > C_5 > \cdots.$$

It remains only to show that any odd-numbered convergent C_{2r-1} is greater than any even-numbered convergent C_{2s}. Since $p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$, upon dividing both sides of the equation by $q_k q_{k-1}$, we obtain

$$C_k - C_{k-1} = \frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{(-1)^{k-1}}{q_k q_{k-1}}.$$

This means that $C_{2j} < C_{2j-1}$. The effect of tying the various inequalities together is that

$$C_{2s} < C_{2s+2r} < C_{2s+2r-1} < C_{2r-1},$$

as desired.

To take an actual example, consider the continued fraction $[2; 3, 2, 5, 2, 4, 2]$. A little calculation gives the convergents

$$C_0 = 2/1,\ C_1 = 7/3,\ C_2 = 16/7,\ C_3 = 87/38,$$
$$C_4 = 190/83,\ C_5 = 847/370,\ C_6 = 1884/823.$$

According to Theorem 13-8, these convergents satisfy the chain of inequalities

$$2 < 16/7 < 190/83 < 1884/823 < 847/370 < 87/38 < 7/3.$$

This is readily visible when the numbers are expressed in decimal notation:

$$2 < 2.28571\cdots < 2.28915\cdots < 2.28918\cdots < 2.28947\cdots < 2.33333\cdots.$$

PROBLEMS 13.3

1. Express each of the rational number below as finite simple continued fractions:
(a) $-19/51$ (b) $187/57$ (c) $71/55$ (d) $118/303$

2. Determine the rational numbers represented by the following simple continued fractions:
(a) $[-2; 2, 4, 6, 8]$ (b) $[4; 2, 1, 3, 1, 2, 4]$ (c) $[0; 1, 2, 3, 4, 3, 2, 1]$

3. If $r = [a_0; a_1, a_2, \ldots, a_n]$, where $r > 1$, show that

$$1/r = [0; a_0, a_1, \ldots, a_n].$$

4. Represent the following simple continued fractions in an equivalent form, but with an odd number of partial denominators:
(a) $[0; 3, 1, 2, 3]$ (b) $[-1; 2, 1, 6, 1]$ (c) $[2; 3, 1, 2, 1, 1, 1]$

5. Compute the convergents of the following simple continued fractions:
(a) $[1; 2, 3, 3, 2, 1]$ (b) $[-3; 1, 1, 1, 1, 3]$ (c) $[0; 2, 4, 1, 8, 2]$

6. (a) If $C_k = p_k/q_k$ is the kth convergent of the simple continued fraction $[1; 2, 3, 4, \ldots, n, n+1]$, show that

$$p_n = np_{n-1} + np_{n-2} + (n-1)p_{n-3} + \cdots + 3p_1 + 2p_0 + (p_0 + 1).$$

[*Hint:* Add the relations $p_0 = 1$, $p_1 = 3$, $p_k = (k+1)p_{k-1} + p_{k-2}$ for $k = 2, \ldots, n$.]

(b) Illustrate part (a) by calculating the numerator p_4 for $[1; 2, 3, 4, 5]$.

7. Evaluate p_k, q_k, and C_k $(k = 0, 1, \ldots, 8)$ for the simple continued fractions below; notice that the convergents provide an approximation to the irrational numbers in parentheses:

(a) $[1; 2, 2, 2, 2, 2, 2, 2, 2]$ $(\sqrt{2})$
(b) $[1; 1, 2, 1, 2, 1, 2, 1, 2]$ $(\sqrt{3})$
(c) $[2; 4, 4, 4, 4, 4, 4, 4, 4]$ $(\sqrt{5})$
(d) $[2; 2, 4, 2, 4, 2, 4, 2, 4]$ $(\sqrt{6})$
(e) $[2; 1, 1, 1, 4, 1, 1, 1, 4]$ $(\sqrt{7})$

8. If $C_k = p_k/q_k$ is the kth convergent of the simple continued fraction $[a_0; a_1, \ldots, a_n]$, establish that

$$q_k \geq 2^{(k-1)/2}, \qquad (2 \leq k \leq n).$$

[*Hint:* Observe that $q_k = a_k q_{k-1} + q_{k-2} \geq 2q_{k-2}$.]

9. Find the simple continued fraction representation of 3.1416, and that of 3.14159.

10. If $C_k = p_k/q_k$ is the kth convergent of the simple continued fraction $[a_0; a_1, \ldots, a_n]$ and $a_0 > 0$, show that

$$p_k/p_{k-1} = [a_k; a_{k-1}, \ldots, a_1, a_0],$$

and

$$q_k/q_{k-1} = [a_k; a_{k-1}, \ldots, a_2, a_1].$$

[*Hint:* In the first case, notice that

$$\begin{aligned} p_k/p_{k-1} &= a_k + (p_{k-2}/p_{k-1}) \\ &= a_k + \frac{1}{(p_{k-1}/p_{k-2})}. \end{aligned}$$
]

11. By means of continued fractions determine the general solutions of each of the following Diophantine equations:
(a) $19x + 51y = 1$; (b) $364x + 227y = 1$;
(c) $18x + 5y = 24$; (d) $158x - 57y = 1$.

12. Verify Theorem 13-8 for the simple continued fraction $[1; 1, 1, 1, 1, 1, 1, 1]$.

13.4 INFINITE CONTINUED FRACTIONS

Up to the point, only finite continued fractions have been considered; and these, when simple, represent rational numbers. One of the main uses of the theory of continued fractions is finding approximate values of irrational numbers. For this, the notion of an infinite continued fraction is necessary.

If $a_0, a_1, a_2, \ldots$ is an infinite sequence of integers, all positive except perhaps for a_0, then the expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}},$$

denoted more simply by $[a_0; a_1, a_2, \ldots]$, is called an *infinite simple continued fraction.* In order to attach a mathematical meaning to this expression, observe that each of the finite continued fractions

$$C_n = [a_0; a_1, a_2, \ldots, a_n] \qquad (n \geq 0)$$

is defined. It seems reasonable therefore to define the value of the infinite continued fraction $[a_0; a_1, a_2, \ldots]$ to be the limit of the sequence of rational numbers C_n, provided of course that this limit exists. In something of an abuse of notation, we shall use $[a_0; a_1, a_2, \ldots]$ to indicate not only the infinite continued fraction, but also its value.

The question of the existence of the above limit is easily settled. For, under our hypothesis, the limit not only exists but is always an irrational number. To see this, observe that formulas previously obtained for finite continued fractions remain valid for infinite continued fractions, since the derivation of these relations did not depend on the finiteness of the fraction. When the upper limits on the indices are removed, Theorem 13-8 tells us that the convergents C_n of $[a_0; a_1, a_2, \ldots]$ satisfy the infinite chain of inequalities

$$C_0 < C_2 < C_4 < \cdots < C_{2n} < \cdots < C_{2n+1} < \cdots < C_5 < C_3 < C_1.$$

Since the even-numbered convergents C_{2n} form a monotonically increasing sequence, bounded above by C_1, they will converge to a limit α which is greater than each C_{2n}. Similarly, the monotonically decreasing sequence of odd-numbered convergents C_{2n+1} is bounded below by C_0 and so has a limit α' which is less than each C_{2n+1}. Let us show that these limits are equal. On the basis of the relation $p_{2n+1}q_{2n} - q_{2n+1}p_{2n} = (-1)^{2n}$ we see that

$$\alpha' - \alpha < C_{2n+1} - C_{2n} = \frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n}}{q_{2n}} = \frac{1}{q_{2n}q_{2n+1}},$$

whence

$$0 \leq |\alpha' - \alpha| < \frac{1}{q_{2n}q_{2n+1}} < \frac{1}{q_{2n}^2}.$$

Since the q_i increase without bound as i becomes large, the right-hand side of this inequality can be made arbitrarily small. If α' and α were not the same, then a contradiction would result (more precisely, $1/q_{2n}^2$ could be made less than the value of $|\alpha' - \alpha|$). Thus, the two sequences of odd- and even-numbered convergents have the same limiting value α, which means that the sequence of convergents C_n has the limit α.

Taking our cue from these remarks, we make the following definition: