DEFINITION 13-3. If $a_0, a_1, a_2, \ldots$ is an infinite sequence of integers, all positive except possibly a_0, then the infinite simple continued fraction $[a_0; a_1, a_2, \ldots]$ has the value $\lim_{n\to\infty} [a_0; a_1, a_2, \ldots, a_n]$.

It should be emphasized again that the adjective "simple" indicates that the partial denominators a_k are all integers; since the only infinite continued fractions to be considered are simple, we shall often omit the term in what follows and call them infinite continued fractions.

Perhaps the most elementary example is afforded by the infinite continued fraction $[1; 1, 1, 1, \ldots]$. Example 13-1 showed that the nth convergent $C_n = [1; 1, 1, \ldots, 1]$, where the integer 1 appears $n+1$ times, is equal to

$$C_n = \frac{u_{n+1}}{u_n} \qquad (n \geq 0),$$

a quotient of successive Fibonacci numbers. If x denotes the value of the continued fraction $[1; 1, 1, 1, \ldots]$, then

$$\begin{aligned} x = \lim_{n\to\infty} C_n = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} &= \lim_{n\to\infty} \frac{u_n + u_{n-1}}{u_n} \\ &= \lim_{n\to\infty} 1 + \cfrac{1}{\cfrac{u_n}{u_{n-1}}} = 1 + \cfrac{1}{\lim\limits_{n\to\infty}\left(\cfrac{u_n}{u_{n-1}}\right)} = 1 + \frac{1}{x}. \end{aligned}$$

This gives rise to the quadratic equation $x^2 - x - 1 = 0$, whose only positive root is $x = (1+\sqrt{5})/2$. Hence,

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \ldots].$$

There is one situation which occurs often enough to merit special terminology. If an infinite continued fraction, such as $[3; 1, 2, 1, 6, 1, 2, 1, 6, \ldots]$, contains a block of partial denominators $b_1, b_2, \ldots, b_n$ which repeats indefinitely, the fraction is called *periodic*. The custom is to write a periodic continued fraction $[a_0; a_1, \ldots, a_m, b_1, \ldots, b_n, b_1, \ldots, b_n, \ldots]$ more compactly as

$$[a_0; a_1, \ldots, a_m, \overline{b_1, \ldots, b_n}],$$

where the bar over $b_1, b_2, \ldots, b_n$ indicates that this block of integers repeats over and over. If $b_1, b_2, \ldots, b_n$ is the smallest block of integers which constantly repeats, we say that $b_1, b_2, \ldots, b_n$ is the *period* of the expansion and that the *length* of the period is n. Thus, for example, $[3; \overline{1, 2, 1, 6}]$ would denote $[3; 1, 2, 1, 6, 1, 2, 1, 6, \ldots]$, a continued fraction whose period 1, 2, 1, 6 has length 4.

We saw earlier that every finite continued fraction is represented by a rational number. Let us now consider the value of an infinite continued fraction.

THEOREM 13-9. *The value of any infinite continued fraction is an irrational number.*

Proof: Suppose that x denotes the value of the infinite continued fraction $[a_0; a_1, a_2, \ldots]$; that is, x is the limit of the sequence of convergents

$$C_n = [a_0; a_1, a_2, \ldots, a_n] = \frac{p_n}{q_n}.$$

Since x lies strictly between the successive convergents C_n and C_{n+1}, we have

$$0 < |x - C_n| < |C_{n+1} - C_n| = \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \frac{1}{q_n q_{n+1}}.$$

With the view to obtaining a contradiction, assume that x is a rational number; say, $x = a/b$, where a and $b > 0$ are integers. Then

$$0 < \left|\frac{a}{b} - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}$$

and so, upon multiplication by the positive number bq_n,

$$0 < |aq_n - bp_n| < \frac{b}{q_{n+1}}.$$

We recall that the q_i increase without bound as i increases. If n is chosen so large that $b < q_{n+1}$, the result is

$$0 < |aq_n - bp_n| < 1.$$

This says that there is a positive integer, namely $|aq_n - bp_n|$, between 0 and 1—an obvious impossibility.

We now ask whether two different infinite continued fractions can represent the same irrational number. Before giving the pertinent result, let us observe that the properties of limits allow us to write an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ as

$$\begin{aligned}
[a_0; a_1, a_2, \ldots] &= \lim_{n\to\infty} [a_0; a_1, \ldots, a_n] \\
&= \lim_{n\to\infty} \left(a_0 + \frac{1}{[a_1; a_2, \ldots, a_n]}\right) \\
&= a_0 + \frac{1}{\lim_{n\to\infty} [a_1; a_2, \ldots, a_n]} \\
&= a_0 + \frac{1}{[a_1; a_2, a_3, \ldots]}.
\end{aligned}$$

Our theorem is stated as:

THEOREM 13-10. *If the infinite continued fractions* $[a_0; a_1, a_2, \ldots]$ *and* $[b_0; b_1, b_2, \ldots]$ *are equal, then* $a_n = b_n$ *for all* $n \geq 0$.

Proof: If $x = [a_0; a_1, a_2, \ldots]$, then $C_0 < x < C_1$, which is the same as saying that $a_0 < x < a_0 + 1/a_1$. Knowing that $a_1 \geq 1$, this produces the inequality $a_0 < x < a_0 + 1$. Hence, $[x] = a_0$, where $[x]$ is the traditional notation for the greatest integer or "bracket" function (page 126).

Now assume that $[a_0; a_1, a_2, \ldots] = x = [b_0; b_1, b_2, \ldots]$ or, to put it in a different form,

$$a_0 + \frac{1}{[a_1; a_2, \ldots]} = x = b_0 + \frac{1}{[b_1; b_2, \ldots]}.$$

By virtue of the conclusion of the first paragraph, we have $a_0 = [x] = b_0$, from which it may then be deduced that $[a_1; a_2, \ldots] = [b_1; b_2, \ldots]$. When the reasoning is repeated, we next conclude that $a_1 = b_1$ and that $[a_2; a_3, \ldots] = [b_2; b_3, \ldots]$. The process continues by mathematical induction, thereby giving $a_n = b_n$ for all $n \geq 0$.

COROLLARY. *Two distinct infinite continued fractions represent two distinct irrational numbers.*

Example 13-4

To determine the unique irrational number represented by the infinite continued fraction $x = [3; 6, \overline{1, 4}]$, let us write $x = [3; 6, y]$, where

$$y = [\overline{1; 4}] = [1; 4, y].$$

Then

$$y = 1 + \frac{1}{4 + 1/y} = 1 + \frac{y}{4y+1} = \frac{5y+1}{4y+1},$$

which leads to the quadratic equation

$$4y^2 - 4y - 1 = 0.$$

Inasmuch as $y > 0$ and this equation has only one positive root, we may infer that

$$y = \frac{1 + \sqrt{2}}{2}.$$

From $x = [3; 6, y]$, we then find that

$$x = 3 + \cfrac{1}{6 + \cfrac{1}{\cfrac{1+\sqrt{2}}{2}}} = \frac{25 + 19\sqrt{2}}{8 + 6\sqrt{2}}$$

$$= \frac{(25 + 19\sqrt{2})(8 - 6\sqrt{2})}{(8 + 6\sqrt{2})(8 - 6\sqrt{2})} = \frac{14 - \sqrt{2}}{4};$$

that is, $[3; 6, \overline{1, 4}] = \frac{14 - \sqrt{2}}{4}$.

Our preceding theorem shows that every infinite continued fraction represents a unique irrational number. Turning matters around, we next establish that any irrational number x_0 can be expanded into an infinite continued fraction $[a_0; a_1, a_2, \ldots]$ which converges to the value x_0. The sequence of integers $a_0, a_1, a_2, \ldots$ is defined as follows: using the bracket function, we first let

$$x_1 = \frac{1}{x_0 - [x_0]}, \quad x_2 = \frac{1}{x_1 - [x_1]}, \quad x_3 = \frac{1}{x_2 - [x_2]}, \quad \cdots$$

and then take

$$a_0 = [x_0], \; a_1 = [x_1], \; a_2 = [x_2], \; a_3 = [x_3], \; \cdots.$$

In general, the a_k are given inductively by

$$a_k = [x_k], \quad x_{k+1} = \frac{1}{x_k - a_k}, \qquad k \geq 0.$$

It is evident that x_{k+1} is irrational, whenever x_k is irrational; and because we are confining ourselves to the case in which x_0 is an irrational number, all x_k are irrational by induction. Thus,

$$0 < x_k - a_k = x_k - [x_k] < 1$$

and we see that

$$x_{k+1} = \frac{1}{x_k - a_k} > 1$$

so that the integer $a_{k+1} = [x_{k+1}] \geq 1$ for all $k \geq 0$. This process therefore leads to an infinite sequence of integers $a_0, a_1, a_2, \ldots$, all positive except perhaps for a_0.