Employing our inductive definition in the form

$$x_k = a_k + \frac{1}{x_{k+1}} \qquad (k \geq 0)$$

we obtain through successive substitution

$$\begin{aligned} x_0 &= a_0 + \frac{1}{x_1} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{x_2}} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{x_3}}} \\ &\vdots \\ &= [a_0; a_1, a_2, \ldots, a_n, x_{n+1}] \end{aligned}$$

for every positive integer n. This makes one suspect—and it is our task to show—that x_0 is the value of the infinite continued fraction $[a_0; a_1, a_2, \ldots]$.

For any fixed n, the first $n+1$ convergents $C_k = p_k/q_k$, $0 \leq k \leq n$, of $[a_0; a_1, a_2, \ldots]$ are the same as the first $n+1$ convergents of the finite continued fraction $[a_0; a_1, a_2, \ldots, a_n, x_{n+1}]$. If we denote the $(n+2)$th convergent of the latter by C'_{n+1}, then the argument used in the proof of Theorem 13-6 to obtain C_{n+1} from C_n by replacing a_n by $a_n + 1/a_{n+1}$ works equally well in the present setting; this enables us to obtain C'_{n+1} from C_n by replacing a_{n+1} by x_{n+1}:

$$\begin{aligned} x_0 = C'_{n+1} &= [a_0; a_1, a_2, \ldots, a_n, x_{n+1}] \\ &= \frac{x_{n+1} p_n + p_{n-1}}{x_{n+1} q_n + q_{n-1}}. \end{aligned}$$

Because of this,

$$\begin{aligned} x_0 - C_n &= \frac{x_{n+1} p_n + p_{n-1}}{x_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{(-1)(p_n q_{n-1} - q_n p_{n-1})}{(x_{n+1} q_n + q_{n-1}) q_n} = \frac{(-1)^n}{(x_{n+1} q_n + q_{n-1}) q_n}, \end{aligned}$$

where the last equality relies on Theorem 13-7. Now $x_{n+1} > a_{n+1}$ and so

$$|x_0 - C_n| = \frac{1}{(x_{n+1} q_n + q_{n-1}) q_n} < \frac{1}{(a_{n+1} q_n + q_{n-1}) q_n} = \frac{1}{q_{n+1} q_n}.$$

Since the integers q_k are increasing, the implication is that

$$x_0 = \lim_{n \to \infty} C_n = [a_0; a_1, a_2, \ldots].$$

Let us sum up our conclusions in

THEOREM 13-11. *Every irrational number has a unique representation as an infinite continued fraction, the representation being obtained from the continued fraction algorithm described above.*

Incidentally, our argument reveals a fact worth recording separately.

COROLLARY. *If p_n/q_n is the nth convergent to the irrational number x, then*

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_{n+1} q_n} < \frac{1}{q_n^2}.$$

We give two examples in illustration of the use of the continued fraction algorithm in finding the representation of a given irrational number as an infinite continued fraction.

Example 13-5

For our first example, consider $x = \sqrt{23} \approx 4.8$. The successive irrational numbers x_k (and therefore the integers $a_k = [x_k]$) can be computed rather easily, with the calculations exhibited below:

$$x_0 = \sqrt{23} = 4 + (\sqrt{23} - 4), \qquad a_0 = 4,$$

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{1}{\sqrt{23} - 4} = \frac{\sqrt{23} + 4}{7} = 1 + \frac{\sqrt{23} - 3}{7}, \qquad a_1 = 1,$$

$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{7}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{2} = 3 + \frac{\sqrt{23} - 3}{2}, \qquad a_2 = 3,$$

$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{2}{\sqrt{23} - 3} = \frac{\sqrt{23} + 3}{7} = 1 + \frac{\sqrt{23} - 4}{7}, \qquad a_3 = 1,$$

$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{7}{\sqrt{23} - 4} = \sqrt{23} + 4 = 8 + (\sqrt{23} - 4), \qquad a_4 = 8.$$

Since $x_5 = x_1$, also $x_6 = x_2$, $x_7 = x_3$, $x_8 = x_4$; then we get $x_9 = x_5 = x_1$, and so on, which means that the block of integers 1, 3, 1, 8 repeats indefinitely. We find that the continued fraction expansion of $\sqrt{23}$ is periodic with the form

$$\sqrt{23} = [4; 1, 3, 1, 8, 1, 3, 1, 8, \ldots] = [4; \overline{1, 3, 1, 8}].$$

Example 13-6

To furnish a second illustration, let us obtain several of the convergents of the continued fraction of the number

$$\pi = 3.141592653 \cdots,$$

defined by the ancient Greeks as the ratio of the circumference of a circle to its diameter. The letter π, from the Greek word *perimetros*, was never employed in antiquity for this ratio; it was Euler's adoption of the symbol in his many popular textbooks that made it widely known and used.

By straightforward calculations, one sees that

$$\begin{aligned}
x_0 &= \pi = 3 + (\pi - 3), & a_0 &= 3,\\
x_1 &= \frac{1}{x_0 - [x_0]} = \frac{1}{0.14159265\ldots} = 7.06251330\ldots, & a_1 &= 7,\\
x_2 &= \frac{1}{x_1 - [x_1]} = \frac{1}{0.06251330\ldots} = 15.99659440\ldots, & a_2 &= 15,\\
x_3 &= \frac{1}{x_2 - [x_2]} = \frac{1}{0.99659440\ldots} = 1.00341723\ldots, & a_3 &= 1,\\
x_4 &= \frac{1}{x_3 - [x_3]} = \frac{1}{0.00341723\ldots} = 292.63724\ldots, & a_4 &= 292,\\
&\vdots
\end{aligned}$$

Thus, the infinite continued fraction for π starts out as

$$\pi = [3; 7, 15, 1, 292, \ldots];$$

but, unlike the case of $\sqrt{23}$ in which all the partial denominators a_n are explicitly known, there is no pattern which gives the complete sequence of a_n. The first five convergents are

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}.$$

As a check on the Corollary to Theorem 3-11, notice that we should have

$$\left|\pi - \frac{22}{7}\right| < \frac{1}{7^2}.$$

Now $314/100 < \pi < 22/7$ and therefore

$$\left|\pi - \frac{22}{7}\right| < \frac{22}{7} - \frac{314}{100} = \frac{1}{7 \cdot 50} < \frac{1}{7^2},$$

as expected.

Unless the irrational number x assumes some very special form, it may be impossible to give the complete continued fraction expansion of x. One can prove, for instance, that the expansion for x becomes ultimately periodic if and only if x is an irrational root of a quadratic equation with integral coefficients; that is, if x takes the form $r+s\sqrt{d}$, where r and $s \neq 0$ are rational numbers and d is a positive integer which is not a perfect square. But among other irrational numbers, there are very few whose representations seem to exhibit any regularity. An exception is another positive constant which has occupied the attention of mathematicians for many centuries, namely

$$e = 2.718281828\ldots,$$

the base of the system of natural logarithms. In 1737, Euler showed that

$$\frac{e-1}{e+1} = [0; 2, 6, 10, 14, 18, \ldots],$$

where the partial denominators form an arithmetic progression, and that

$$\frac{e^2-1}{e^2+1} = [0; 1, 3, 5, 7, 9, \ldots].$$

The continued fraction representation of e itself (also found by Euler) is a bit more complicated, yet still has a pattern:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots],$$

with the even integers subsequently occurring in order and separated by two 1's. With regard to the symbol e, its use is also original with Euler and it appeared in print for the first time in one of his textbooks.

In the introduction to analysis, it is usually demonstrated that e can be defined by the infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

If the reader is willing to accept this fact, then Euler's proof of the irrationality of e can be given very quickly: Suppose to the contrary that e is rational, say $e = a/b$, where a and b are positive integers. Then for $n > b$ and also $n > 1$, the number

$$N = n!\left(e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right) = n!\left(\frac{a}{b} - 1 - \frac{1}{1!} - \frac{1}{2!} - \cdots - \frac{1}{n!}\right)$$