is a positive integer. When e is replaced by its series expansion, this becomes

$$N=\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots$$
$$<\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\cdots$$
$$=\frac{1}{n+1}+\left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)+\cdots=\frac{2}{n+1}<1.$$

Since the inequality $0<N<1$ is impossible for an integer, e must be irrational. The exact nature of the number π offers greater difficulties; J. H. Lambert (1728–1777), in 1761, communicated to the Berlin Academy an essentially rigorous proof of the irrationality of π.

Given an irrational number x, a natural question is to ask how closely, or with what degree of accuracy, it can be approximated by rational numbers. One way of approaching the problem is to consider all rational numbers with a fixed denominator $b>0$. Since x lies between two such rational numbers, say $c/b<x<(c+1)/b$, it follows that

$$\left|x-\frac{c}{b}\right|<\frac{1}{b}.$$

Better yet, we can write

$$\left|x-\frac{a}{b}\right|<\frac{1}{2b},$$

where $a=c$ or $a=c+1$, whichever choice may be appropriate. The continued fraction process permitted us to prove a result which considerably strengthens the last-written inequality, namely: given any irrational number x, there exist infinitely many rational numbers a/b in lowest terms which satisfy

$$\left|x-\frac{a}{b}\right|<\frac{1}{b^2}.$$

In fact, by the corollary to Theorem 13-11, any of the convergents p_n/q_n of the continued fraction expansion of x can play the role of the rational number a/b. The forthcoming theorem asserts that the convergents p_n/q_n have the property of being the best approximations, in the sense of giving the closest approximation to x among all rational numbers a/b with denominators q_n or less.

For clarity, the technical core of the theorem is placed in the following lemma.

LEMMA. *Let p_n/q_n be the nth convergent of the continued fraction representing the irrational number x. If a and b are integers, with $1 \leq b < q_{n+1}$, then*

$$|q_n x - p_n| \leq |bx - a|.$$

Proof: Consider the system of equations

$$p_n\alpha + p_{n+1}\beta = a,$$
$$q_n\alpha + q_{n+1}\beta = b.$$

The determinant of the coefficients being $p_n q_{n+1} - q_n p_{n+1} = (-1)^{n+1}$, the system has the unique integral solution

$$\alpha = (-1)^{n+1}(aq_{n+1} - bp_{n+1}),$$
$$\beta = (-1)^{n+1}(bp_n - aq_n).$$

It is well to notice that $\alpha \neq 0$. In fact, $\alpha = 0$ yields $aq_{n+1} = bp_{n+1}$ and, because $\gcd(p_{n+1}, q_{n+1}) = 1$, this means that $q_{n+1} \mid b$ or $b \geq q_{n+1}$, contrary to hypothesis. In the event that $\beta = 0$, the inequality stated in the lemma is clearly true. For $\beta = 0$ leads to $a = p_n\alpha$, $b = q_n\alpha$ and, as a result,

$$|bx - a| = |\alpha|\,|q_n x - p_n| \geq |q_n x - p_n|.$$

Thus, there is no harm in assuming hereafter that $\beta \neq 0$.

When $\beta \neq 0$, we argue that α and β must have opposite signs. If $\beta < 0$, then the equation $q_n\alpha = b - q_{n+1}\beta$ indicates that $q_n\alpha > 0$ and, in turn, $\alpha > 0$. On the other hand if $\beta > 0$, then $b < q_{n+1}$ implies that $b < \beta q_{n+1}$ and therefore $\alpha q_n = b - q_{n+1}\beta < 0$; this makes $\alpha < 0$. We also infer that, because x stands between the consecutive convergents p_n/q_n and p_{n+1}/q_{n+1},

$$q_n x - p_n \qquad \text{and} \qquad q_{n+1}x - p_{n+1}$$

will have opposite signs. The point of this reasoning is that the numbers

$$\alpha(q_n x - p_n) \qquad \text{and} \qquad \beta(q_{n+1}x - p_{n+1})$$

must have the same sign; in consequence, the absolute value of their sum equals the sum of their separate absolute values. It is this crucial fact that allows us to complete the proof quickly:

$$\begin{aligned} |bx - a| &= |(q_n\alpha + q_{n+1}\beta)x - (p_n\alpha + p_{n+1}\beta)| \\ &= |\alpha(q_n x - p_n) + \beta(q_{n+1}x - p_{n+1})| \\ &= |\alpha|\,|q_n x - p_n| + |\beta|\,|q_{n+1}x - p_{n+1}| \\ &> |\alpha|\,|q_n x - p_n| \geq |q_n x - p_n|, \end{aligned}$$

which is the desired inequality.

The convergents p_n/q_n are best approximations to the irrational number x in that every other rational number with the same or smaller denominator differs from x by a greater amount.

THEOREM 13-12. *If* $1 \leq b \leq q_n$, *the rational number* a/b *satisfies*

$$\left|x - \frac{p_n}{q_n}\right| \leq \left|x - \frac{a}{b}\right|.$$

Proof: Were it to happen that

$$\left|x - \frac{p_n}{q_n}\right| > \left|x - \frac{a}{b}\right|,$$

then

$$|q_n x - p_n| = q_n\left|x - \frac{p_n}{q_n}\right| > b\left|x - \frac{a}{b}\right| = |bx - a|,$$

violating the conclusion of the lemma.

Historians of mathematics have focused considerable attention on the attempts of early societies to arrive at an approximation to π, perhaps because the increasing accuracy of the results seems to offer a measure of the mathematical skills of different cultures. The first recorded scientific effort to evaluate π appeared in the *Measurement of a Circle* by the great mathematician of ancient Syracuse, Archimedes (287–212 B.C.). Substantially, his method for finding the value of π was to inscribe and circumscribe regular polygons about a circle, determine their perimeters, and use these as lower and upper bounds on the circumference. By this means, and using a polygon of 96 sides, he obtained the two approximations in the inequality $223/71 < \pi < 22/7$.

Theorem 13-12 provides insight into why 22/7, the so-called "Archimedian value of π," was used so frequently in place of π; there is no fraction, given in lowest terms, with smaller denominator which furnishes a better approximation. While

$$\left|\pi - \frac{22}{7}\right| \approx 0.0012645 \qquad \text{and} \qquad \left|\pi - \frac{223}{71}\right| \approx 0.0007476,$$

Archimedes' value of 223/71, which is not a convergent of π, has a denominator exceeding $q_2 = 7$. Our theorem tells us that 333/106 (a ratio for π employed in Europe in the 16th century) will approximate π

more closely than any rational number with denominator less than or equal to 106; indeed,

$$\left|\pi - \frac{333}{106}\right| \approx 0.0000832.$$

Due to the size of $q_4 = 33102$, the convergent $p_3/q_3 = 355/113$ allows one to approximate π with a striking degree of accuracy; from the corollary to Theorem 13-11, we have

$$\left|\pi - \frac{355}{113}\right| < \frac{1}{113 \cdot 33102} < \frac{3}{10^7}.$$

The noteworthy ratio of 355/113 was known to the early Chinese mathematician Tsu Chung-chih (430–501); by some reasoning not stated in his works, he gave 22/7 as an "inaccurate value" of π and 355/113 as the "accurate value." The accuracy of the latter ratio was not equalled in Europe until the end of the 16th century, when Adriaen Anthoniszoon rediscovered the identical value.

This is a convenient place to record a theorem which says that any "close" (in a suitable sense) rational approximation to x must be a convergent to x. There would be a certain neatness to the theory if

$$\left|x - \frac{a}{b}\right| < \frac{1}{b^2}$$

implied that $a/b = p_n/q_n$ for some n; while this is too much to hope for, a slightly sharper inequality guarantees the same conclusion.

THEOREM 13-13. *Let x be an irrational number. If the rational number a/b, where $b \geq 1$ and* $\gcd(a, b) = 1$, *satisfies*

$$\left|x - \frac{a}{b}\right| < \frac{1}{2b^2}$$

then a/b is one of the convergents p_n/q_n in the continued fraction representation of x.

Proof: Assume that a/b is not a convergent of x. Knowing that the q_k form an increasing sequence, we see that there is a unique integer n for which $q_n \leq b < q_{n+1}$. For this n, the last lemma gives the first inequality in the chain

$$|q_n x - p_n| \leq |bx - a| = b\left|x - \frac{a}{b}\right| < \frac{1}{2b},$$