which may be recast as

$$\left|x-\frac{p_n}{q_n}\right|<\frac{1}{2bq_n}.$$

In view of the supposition that $a/b \neq p_n/q_n$, the difference $bp_n - aq_n$ is a nonzero integer, whence $1 \leq |bp_n - aq_n|$. We are able to conclude at once that

$$\frac{1}{bq_n} \leq \left|\frac{bp_n - aq_n}{bq_n}\right| = \left|\frac{p_n}{q_n}-\frac{a}{b}\right| \leq \left|\frac{p_n}{q_n}-x\right| + \left|x-\frac{a}{b}\right| < \frac{1}{2bq_n}+\frac{1}{2b^2}.$$

This produces the contradiction $b < q_n$, ending the proof.

PROBLEMS 13.4

1. Evaluate each of the following infinite simple continued fractions:

   (a) $[\overline{2; 3}]$ (b) $[0; \overline{1, 2, 3}]$

   (c) $[2; \overline{1, 2, 1}]$ (d) $[1; 2, \overline{3, 1}]$

   (e) $[1; 2, 1, 2, \overline{12}]$

2. Prove that if the irrational number $x > 1$ is represented by the infinite continued fraction $[a_0; a_1, a_2, \ldots]$, then $1/x$ has the expansion $[0; a_0, a_1, a_2, \ldots]$. Use this fact to find the value of $[0; 1, 1, 1, \ldots] = [0; \overline{1}]$.

3. Evaluate $[1; 2, \overline{1}]$ and $[1; 2, 3, \overline{1}]$.

4. Determine the infinite continued fraction representation of each irrational number below:

   (a) $\sqrt{5}$ (b) $\sqrt{7}$

   (c) $\dfrac{1+\sqrt{13}}{2}$ (d) $\dfrac{5+\sqrt{37}}{4}$

   (e) $\dfrac{11+\sqrt{30}}{13}$

5. (a) For any positive integer n, show that $\sqrt{n^2+1} = [n; \overline{2n}]$, $\sqrt{n^2+2} = [n; \overline{n, 2n}]$ and $\sqrt{n^2+2n} = [n; \overline{1, 2n}]$. [*Hint:* Notice that

   $$n+\sqrt{n^2+1} = 2n + (\sqrt{n^2+1}-n) = 2n + \frac{1}{n+\sqrt{n^2+1}}.\Bigg]$$

   (b) Use part (*a*) to obtain the continued fraction representations of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{15}$ and $\sqrt{37}$.

6. Among the convergents of $\sqrt{15}$, find a rational number which will approximate $\sqrt{15}$ with accuracy to four decimal places.

7. (a) Find a rational approximation to $e=[2; 1, 2, 1, 1, 4, 1, 1, 6, \ldots]$ which is correct to 4 decimal places.
   (b) If a and b are positive integers, show that the inequality $e < a/b < 87/32$ implies that $b \geq 39$.
8. Prove that of any two consecutive convergents of the irrational number x, at least one, a/b, satisfies the inequality

$$\left|x-\frac{a}{b}\right|<\frac{1}{2b^2}.$$

[*Hint:* Since x lies between any two consecutive convergents,

$$\frac{1}{q_n q_{n+1}}=\left|\frac{p_{n+1}}{q_{n+1}}-\frac{p_n}{q_n}\right|=\left|x-\frac{p_{n+1}}{q_{n+1}}\right|+\left|x-\frac{p_n}{q_n}\right|.$$

Now argue by contradiction.]

9. Given the infinite continued fraction $[1; 3, 1, 5, 1, 7, 1, 9, \ldots]$, find the best rational approximation a/b with
   (a) denominator $b < 25$; (b) denominator $b < 225$.
10. First show that $|(1+\sqrt{10})/3-18/13| < 1/(2 \cdot 13^2)$; and then verify that $18/13$ is a convergent of $(1+\sqrt{10})/3$.
11. A famous theorem of A. Hurwitz (1891) says that for any irrational number x, there exist infinitely many rational numbers a/b such that

$$\left|x-\frac{a}{b}\right|<\frac{1}{\sqrt{5}b^2}.$$

Taking $x=\pi$, obtain three rational numbers satisfying this inequality.

12. Assume that the continued fraction representation for the irrational number x ultimately becomes periodic. Mimic the method used in Example 13-4 to prove that x is of the form $r+s\sqrt{d}$, where r and $s \neq 0$ are rational numbers and $d>0$ is a nonsquare integer.
13. Let x be an irrational number with convergents p_n/q_n. For every $n \geq 0$, verify that
    (a) $1/2q_n q_{n+1} < |x-p_n/q_n| < 1/q_n q_{n+1}$;
    (b) the convergents are successively closer to x in the sense that

$$\left|x-\frac{p_n}{q_n}\right|<\left|x-\frac{p_{n-1}}{q_{n-1}}\right|.$$

[*Hint:* Rewrite the relation

$$x=\frac{x_{n+1}q_n+p_{n-1}}{x_{n+1}q_n+q_{n-1}}$$

as $x_{n+1}(xq_n-p_n)=-q_{n-1}(x-p_{n-1}/q_{n-1})$.]

13.5 PELL'S EQUATION

What little action Fermat took to publicize his discoveries came in the form of challenges to other mathematicians. Perhaps he hoped in this way to convince them that his new style of number theory was worth pursuing. In January of 1657, Fermat proposed to the European mathematical community—thinking probably in the first place of John Wallis, England's most renowned practitioner before Newton—a pair of problems:

1. Find a cube which, when increased by the sum of its proper divisors, becomes a square; for example, $7^3 + (1 + 7 + 7^2) = 20^2$.
2. Find a square which, when increased by the sum of its proper divisors, becomes a cube.

On hearing of the contest, Fermat's favorite correspondent, Bernhard Frénicle de Bessy, quickly supplied a number of answers to the first problem; typical of these is $(2 \cdot 3 \cdot 5 \cdot 13 \cdot 41 \cdot 47)^3$, which when increased by the sum of its proper divisors becomes $(2^7 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 29)^2$. While Frénicle advanced to solutions in still larger composite numbers, Wallis dismissed the problems as not worth his effort, writing, "Whatever the details of the matter, it finds me too absorbed by numerous occupations for me to be able to devote my attention to it immediately; but I can make at this moment this response: the number 1 in and of itself satisfies both demands." Barely concealing his disappointment, Frénicle expressed astonishment that a mathematician as experienced as Wallis would have made only the trivial response when, in view of Fermat's stature, he should have sensed the problem's greater depths.

Fermat's interest, indeed, lay in general methods, not in the wearying computation of isolated cases. Both Frénicle and Wallis overlooked the theoretical aspect that the challenge-problems were meant to reveal on careful analysis. While the phrasing was not entirely precise, it seems clear that Fermat had intended the first of his queries to be solved for cubes of prime numbers. To put it otherwise, the problem called for finding all integral solutions of the equation

$$1 + x + x^2 + x^3 = y^2,$$

or equivalently

$$(1 + x)(1 + x^2) = y^2,$$

where x is an odd integer. Since 2 is the only prime which divides both factors on the left-hand side of this equation, it may be written as

$$ab = \left(\frac{y}{2}\right)^2, \qquad \gcd(a, b) = 1.$$

But if the product of two relatively prime integers is a perfect square, then each of them must be a square; hence, $a = u^2$, $b = v^2$ for some u and v, so that

$$1 + x = 2a = 2u^2, \qquad 1 + x^2 = 2b = 2v^2.$$

This means that any integer x which satisfies Fermat's first problem must be a solution of the pair of equations

$$x = 2u^2 - 1, \qquad x^2 = 2v^2 - 1,$$

the second being a particular case of the equation $x^2 = dy^2 \pm 1$.

In February, 1657, Fermat issued his Second Challenge, dealing directly with the theoretical point at issue: Find a number y which will make dy^2+1 a perfect square, where d is a positive integer which is not a square; for example, $3 \cdot 1^2+1=2^2$ and $5 \cdot 4^2+1=9^2$. If, said Fermat, a general rule cannot be obtained, find the smallest values of y which will satisfy the equations $61y^2 + 1 = x^2$; or $109y^2 + 1 = x^2$. Frénicle proceeded to calculate the smallest positive solutions of $x^2 - dy^2 = 1$ for all permissible values of d up to 150 and suggested that Wallis extend the table to $d = 200$ or at least solve $x^2 - 151y^2 = 1$ and $x^2 - 313y^2 = 1$, hinting that the second equation might be beyond Wallis' ability. In reply, Wallis' patron Lord William Brouncker of Ireland stated that it had only taken him an hour or so to discover that

$$(126862368)^2 - 313(7170685)^2 = -1$$

and so $y = 2 \cdot 7170685 \cdot 126862368$ gives the desired solution to $x^2 - 313y^2 = 1$; Wallis solved the other concrete case, furnishing

$$(1728148040)^2 - 151(140634693)^2 = 1.$$

The size of these numbers in comparison with those arising from other values of d suggests that Fermat was in possession of a complete solution to the problem, but this was never disclosed (later, he affirmed that his method of infinite descent had been used with success to show the existence of an infinitude of solutions of $x^2 - dy^2 = 1$). Brouncker,