under the mistaken impression that rational and not necessarily integral values were allowed, had no difficulty in supplying an answer; he simply divided the relation

$$(r^2 + d)^2 - d(2r)^2 = (r^2 - d)^2$$

by the quantity $(r^2 - d)^2$ to arrive at the solution

$$x = \frac{r^2 + d}{r^2 - d}, \qquad y = \frac{2r}{r^2 - d}$$

where $r \neq d$ is an arbitrary rational number. This, needless to say, was rejected by Fermat, who wrote that "solutions in fractions, which can be given at once from the merest elements of arithmetic, do not satisfy me." Now informed of all the conditions of the challenge, Brouncker and Wallis jointly devised a tentative method for solving $x^2 - dy^2 = 1$ in integers, without being able to give a proof that it will always work. Apparently the honors rested with Brouncker, for Wallis congratulated Brouncker with some pride that he had "preserved untarnished the fame that Englishmen have won in former times with Frenchmen."

After having said all this, we should record that Fermat's well-directed effort to institute a new tradition in arithmetic through a mathematical joust was largely a failure. Save for Frénicle, who lacked the talent to vie in intellectual combat with Fermat, number theory had no special appeal to any of his contemporaries. The subject was permitted to fall into disuse, until Euler, after the lapse of nearly a century, picked up where Fermat had left off. Both Euler and Lagrange contributed to the resolution of the celebrated problem of 1657. By converting $\sqrt{d}$ into an infinite continued fraction, Euler (1759) invented a procedure for obtaining the smallest integral solution of $x^2 - dy^2 = 1$, but he failed to show that the process leads to a solution other than $x = 1, y = 0$. It was left to Lagrange to clear up this matter. Completing the theory left unfinished by Euler, Lagrange in 1768 published the first rigorous proof that all solutions arise through the continued fraction expansion of $\sqrt{d}$.

As a result of a mistaken reference, the central point of contention, the equation $x^2 - dy^2 = 1$, has gone into the literature with the title "Pell's equation." The erroneous attribution of its solution to the English mathematician John Pell (1611–1685), who had little to do with the problem, was an oversight on Euler's part. On a cursory reading of Wallis' *Opera Mathematica* (1693), in which Brouncker's method of solving the equation is set forth as well as information as to Pell's work on diophantine analysis, Euler must have confused their contributions.

By all rights we should call $x^2-dy^2=1$ "Fermat's equation," for he was the first to deal with it systematically. While the historical error has long been recognized, Pell's name is the one that is indelibly attached to the equation.

Whatever the integral value of d, the equation $x^2-dy^2=1$ is satisfied trivially by $x=\pm 1, y=0$. If $d<-1$, then $x^2-dy^2\geq 1$ (except when $x=y=0$) so that these exhaust the solutions; when $d=-1$, two more solutions occur, namely $x=0, y=\pm 1$. The case in which d is a perfect square is easily dismissed. For if $d=n^2$ for some n, then $x^2-dy^2=1$ can be written in the form

$$(x+ny)(x-ny)=1$$

which is possible if and only if $x+ny=x-ny=\pm 1$; it follows that

$$x=\frac{(x+ny)+(x-ny)}{2}=\pm 1$$

and the equation has no solutions apart from the trivial ones $x=\pm 1, y=0$.

From now on, we shall restrict our investigation of the Pell equation $x^2-dy^2=1$ to the only interesting situation, that where d is a positive integer which is not a square. Let us say that a solution x, y of this equation is a *positive solution* provided both x and y are positive. Since solutions beyond those with $y=0$ can be arranged in sets of four by combinations of signs $\pm x$, $\pm y$, it is clear that all solutions will be known once all positive solutions have been found. For this reason, we seek only positive solutions of $x^2-dy^2=1$.

The result which provides us with a starting point asserts that any pair of positive integers satisfying Pell's equation can be obtained from the continued fraction representing the irrational number $\sqrt{d}$.

THEOREM 13-14. *If p, q is a positive solution of $x^2-dy^2=1$, then p/q is a convergent of the continued fraction expansion of $\sqrt{d}$.*

Proof: In light of the hypothesis that $p^2-dq^2=1$, we have

$$(p-q\sqrt{d})(p+q\sqrt{d})=1$$

implying that $p>q\sqrt{d}$ as well as that

$$\frac{p}{q}-\sqrt{d}=\frac{1}{q(p+q\sqrt{d})}.$$

As a result,

$$0<\frac{p}{q}-\sqrt{d}<\frac{\sqrt{d}}{q(q\sqrt{d}+q\sqrt{d})}=\frac{\sqrt{d}}{2q^2\sqrt{d}}=\frac{1}{2q^2}.$$

A direct appeal to Theorem 13-13 indicates the p/q must be a convergent of $\sqrt{d}$.

In general, the converse of the preceding theorem is false: not all of the convergents p_n/q_n of $\sqrt{d}$ supply solutions to $x^2-dy^2=1$. Nonetheless, we can say something about the size of the values taken on by the sequence $p_n^2-dq_n^2$.

THEOREM 13-15. *If p/q is a convergent of the continued fraction expansion of $\sqrt{d}$, then $x=p, y=q$ is a solution of one of the equations*

$$x^2-dy^2=k,$$

where $|k|<1+2\sqrt{d}$.

Proof: If p/q is a convergent of $\sqrt{d}$, then the corollary to Theorem 13-11 guarantees that

$$\left|\sqrt{d}-\frac{p}{q}\right|<\frac{1}{q^2}$$

and therefore

$$|p-q\sqrt{d}|<\frac{1}{q}.$$

This being so, we have

$$|p+q\sqrt{d}|=|(p-q\sqrt{d})+2q\sqrt{d}|<\frac{1}{q}+2q\sqrt{d}<(1+2\sqrt{d})q.$$

These two inequalities combine to yield

$$|p^2-dq^2|=|p-q\sqrt{d}|\,|p+q\sqrt{d}|<\frac{1}{q}(1+2\sqrt{d})q=1+2\sqrt{d},$$

which is precisely what was to be proved.

In illustration let us take the case of $d=7$. Using the continued fraction expansion $\sqrt{7}=[2;\overline{1,1,1,4}]$, the first few convergents of $\sqrt{7}$ are determined to be

$$2/1,\ 3/1,\ 5/2,\ 8/3,\ \ldots.$$

Running through the calculations of $p_n^2-7q_n^2$, we find that

$$2^2-7\cdot 1^2=-3,\quad 3^2-7\cdot 1^2=2,\quad 5^2-7\cdot 2^2=-3,\quad 8^2-7\cdot 3^2=1,$$

whence $x=8, y=3$ provides a positive solution of the equation $x^2-7y^2=1$.

While a rather elaborate study can be made of periodic continued fractions, it is not our intention to explore this area at any length. The reader may have noticed already that in the examples considered so far, the continued fraction expansions of $\sqrt{d}$ all took the form

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_n}];$$

that is, the periodic part starts after one term, this initial term being $[\sqrt{d}]$. It is also true that the last term a_n of the period is always equal to $2a_0$ and that the period, with the last term excluded, is symmetrical (the symmetrical part may or may not have a middle term). This is typical of the general situation. Without entering into the details of proof, let us simply record the fact: if d is a positive integer which is not a perfect square, then the continued fraction expansion of $\sqrt{d}$ necessarily has the form

$$\sqrt{d} = [a_0; \overline{a_1, a_2, a_3, \ldots, a_3, a_2, a_1, 2a_0}].$$

In the case in which $d = 19$, for instance, the expansion is

$$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$$

while $d = 73$ gives

$$\sqrt{73} = [8; \overline{1, 1, 5, 5, 1, 1, 16}].$$

Among all $d < 100$, the longest period is that of $\sqrt{94}$ which has sixteen terms:

$$\sqrt{94} = [9; \overline{1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18}].$$

The accompanying table lists the continued fraction expansions of $\sqrt{d}$, where d is a nonsquare integer between 2 and 40.

$\sqrt{2} = [1; \overline{2}]$	$\sqrt{17} = [4; \overline{8}]$	$\sqrt{29} = [5; \overline{2, 1, 1, 2, 10}]$
$\sqrt{3} = [1; \overline{1, 2}]$	$\sqrt{18} = [4; \overline{4, 8}]$	$\sqrt{30} = [5; \overline{2, 10}]$
$\sqrt{5} = [2; \overline{4}]$	$\sqrt{19} = [4; \overline{2, 1, 3, 1, 2, 8}]$	$\sqrt{31} = [5; \overline{1, 1, 3, 5, 3, 1, 1, 10}]$
$\sqrt{6} = [2; \overline{2, 4}]$	$\sqrt{20} = [4; \overline{2, 8}]$	$\sqrt{32} = [5; \overline{1, 1, 1, 10}]$
$\sqrt{7} = [2; \overline{1, 1, 1, 4}]$	$\sqrt{21} = [4; \overline{1, 3, 1, 8}]$	$\sqrt{33} = [5; \overline{1, 2, 1, 10}]$
$\sqrt{8} = [2; \overline{1, 4}]$	$\sqrt{22} = [4; \overline{1, 2, 4, 2, 1, 8}]$	$\sqrt{34} = [5; \overline{1, 4, 1, 10}]$
$\sqrt{10} = [3; \overline{6}]$	$\sqrt{23} = [4; \overline{1, 3, 1, 8}]$	$\sqrt{35} = [5; \overline{1, 10}]$
$\sqrt{11} = [3; \overline{3, 6}]$	$\sqrt{24} = [4; \overline{1, 8}]$	$\sqrt{37} = [6; \overline{12}]$
$\sqrt{12} = [3; \overline{2, 6}]$	$\sqrt{26} = [5; \overline{10}]$	$\sqrt{38} = [6; \overline{6, 12}]$
$\sqrt{13} = [3; \overline{1, 1, 1, 6}]$	$\sqrt{27} = [5; \overline{5, 10}]$	$\sqrt{39} = [6; \overline{4, 12}]$
$\sqrt{14} = [3; \overline{1, 2, 1, 6}]$	$\sqrt{28} = [5; \overline{3, 2, 3, 10}]$	$\sqrt{40} = [6; \overline{3, 12}]$
$\sqrt{15} = [3; \overline{1, 6}]$		